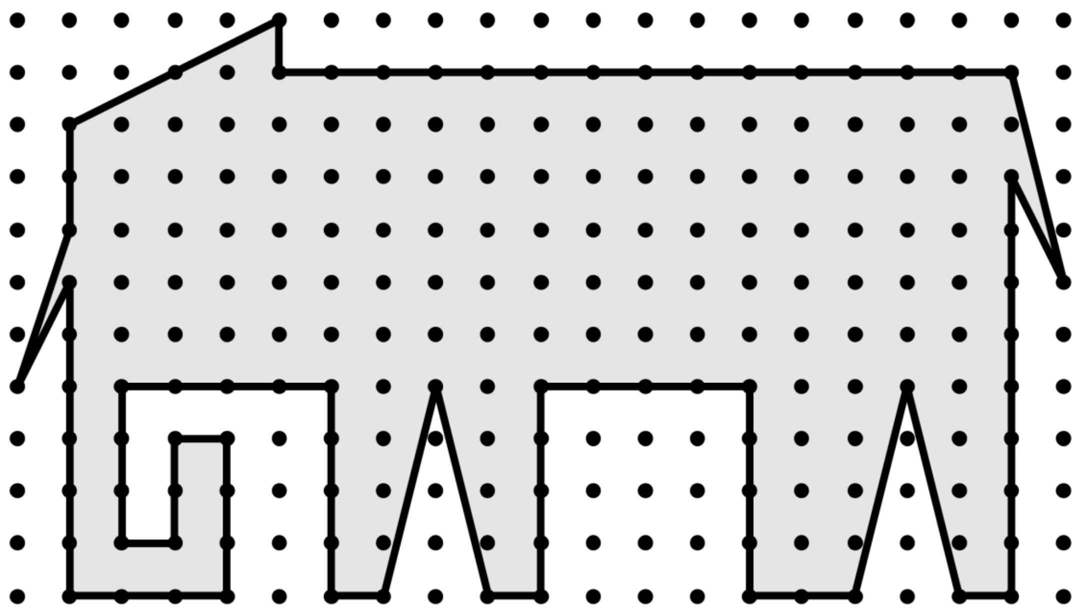
Counting Areas

Task 1:

King Bahubali loved elephants so much that he kept a herd of them. In fact, he planted his coconut garden in such a way that it looked like an elephant when viewed from his terrace!
But the elephants would walk around the garden and destroy it. So, the king put a fence around the garden to keep the elephants away as seen in the figure given below. The trees were planted on a square grid, with one tree at each grid point, to provide sufficient space for each tree. If the king's grounds were 20 units long and 11 units wide, can you find the area available for the elephants to roam, by just counting the coconut trees?

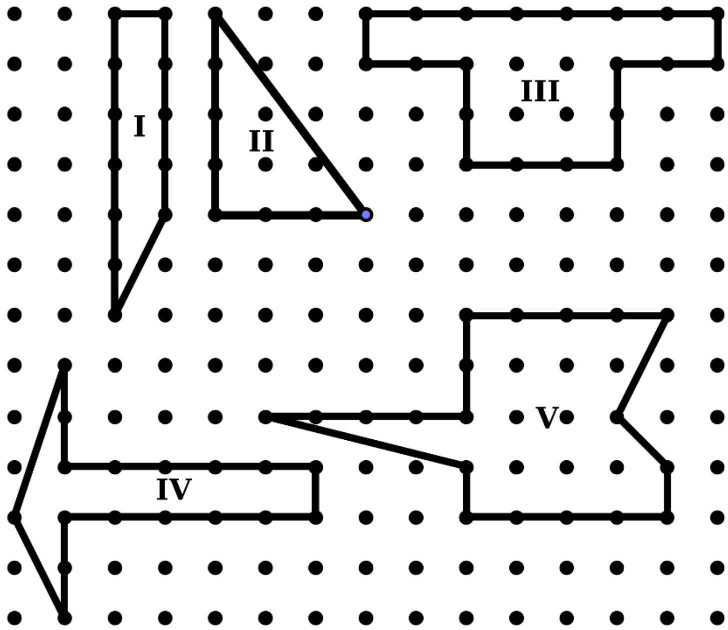
If you cannot solve it now, go ahead with the remaining tasks, and you will be able to do this at the end of the tasks!!



Task 2:

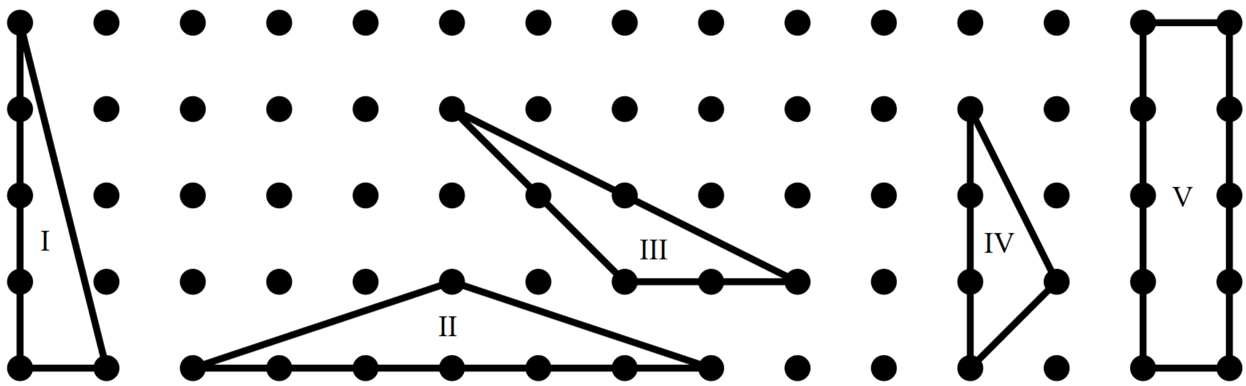
Given below are some figures. Find the area of each and complete the given table.

Figure	Area in Sq Units
I	
II	
III	
IV	
V	



Task 3: Some more figures!

a) Find the area of the following figures.



Also, count the number of grid-points on the boundary of each figure, and fill the table below.

Figures	Area in Square Units	Number of grid-points on the boundary (B)
I		
II		
III		
IV		
V		

b) Do you see any relation between the area of the triangle and the number of grid-points on its boundary?

c) Does the same relation hold for figures I to V in Task 2? If not, for which ones does the relation hold?

I) The relation holds for figures _______________. (Write the number of the figure.)

ii) The relation does not hold for figures ______________. (Write the number of the figure.)

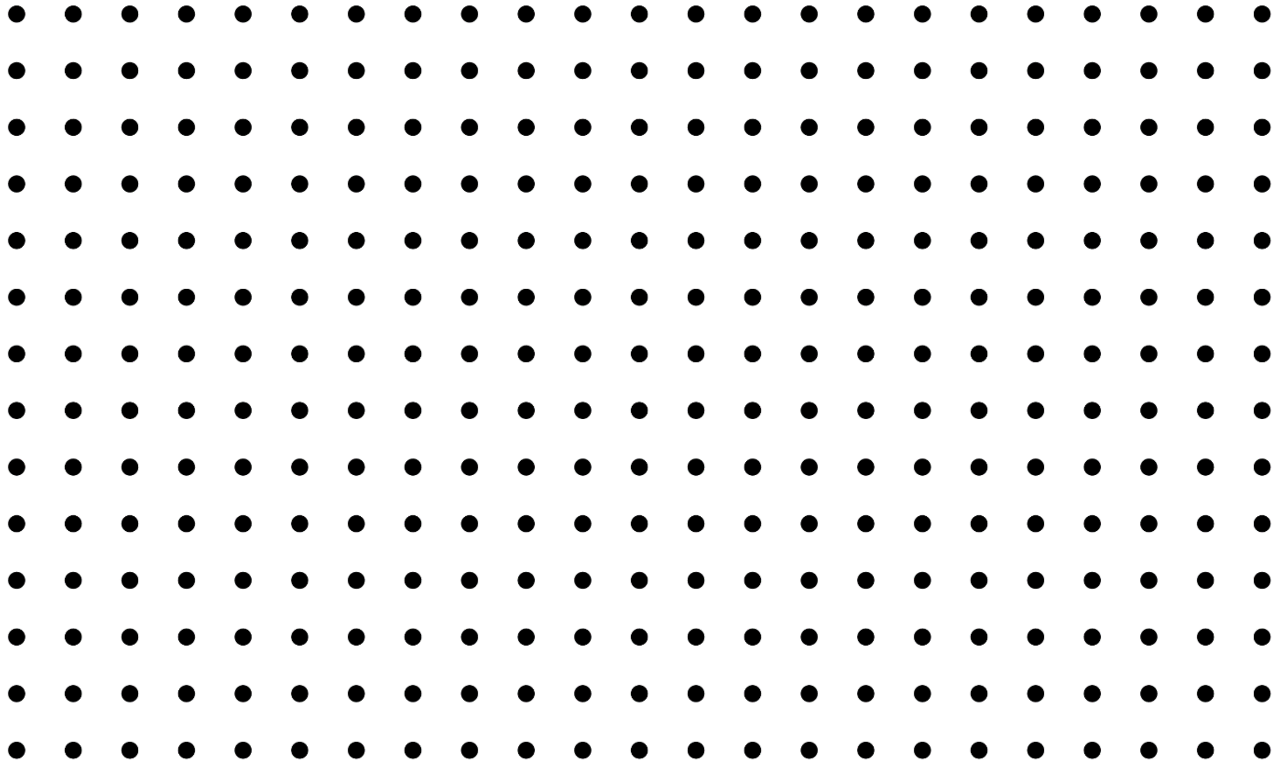
Task 4: Finding the expression!

a) In Task 3c), how are the figures in i) different from the figures in ii)? What property distinguishes figures in i) from figures in ii)?

b) How would you modify the relation in Task 3b) such that it holds for all figures?

Task 5: Making some more figures

Draw five more figures on the grid provided below and check if the relation holds for these figures as well. Are you sure that it will hold for any figure that you may draw? What are the properties common to the figures for which this relation holds?



Did your relation hold for the figures you drew on the above grid?

We have looked at some polygons and found an expression for their area by just counting the boundary and the interior points.

Now let us look at some special polygons and prove that this expression holds for them too.

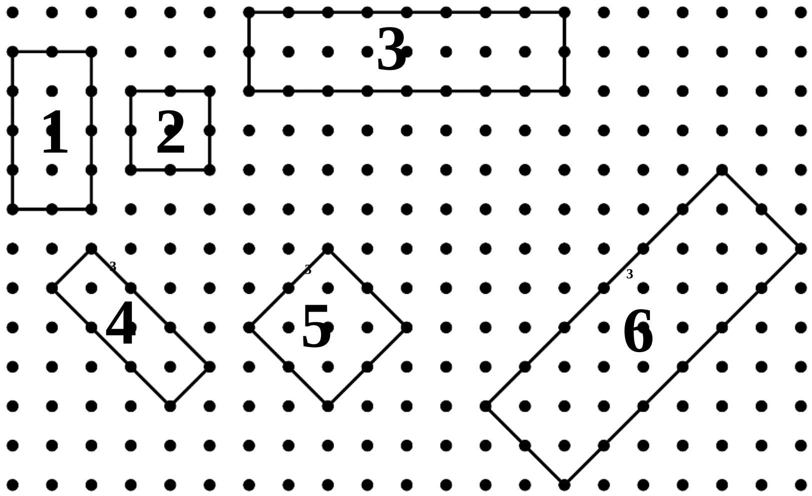
Task 6: Special cases!

Some special polygons:

In the upcoming calculations, we are going to look at some very special type of quadrilaterals, namely straight squares, and rectangles.

What do we mean by straight squares or rectangles?

Look at the rectangles given below:



In the figure: We will call rectangles 1, 2, and 3 as straight rectangles and rectangles 4, 5 and 6 as slanted rectangles. Note that Rectangle 2 is also a straight square and Rectangle 5 is a slanted square.

a) For a straight square of side *m* units

The number of grid-points in the interior (I) is ___________

The number of grid-points on the boundary (B)is ___________.

$I + \frac{B}{2} - 1$ = ___________.

How is the expression $I + \frac{B}{2} - 1$ related to the expression for the area of the square?

b) For a straight rectangle of length *m* units and breadth *n* units,

The number of grid-points in the interior (I) = ______________

The number of grid-points on the boundary (B) = ___________

$I + \frac{B}{2} - 1$ = ___________

How is the expression $I + \frac{B}{2} - 1$ related to the expression for the area of the rectangle?

For figure A, with I grid-points in its interior and B grid-points on its boundary, let us call

$I + \frac{B}{2} - 1$ as **Pick(A).**

Then for a straight rectangle and a straight square, we saw that

Pick(A) = Area(A)

This is called Pick's Theorem applied to a straight square or a rectangle.

So, we have proved Pick's theorem for a very special class of figures namely straight squares and straight rectangles. We now will go on to see if Pick's Theorem is true for all figures on the grid paper. But before that, let us go back to Task 1!

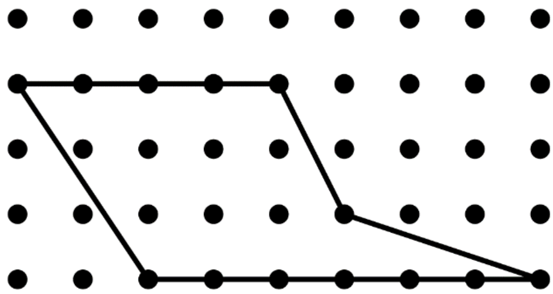
c) Complete Task 1.

Task 7: What about any polygons?

We have proved that Pick's theorem holds for any straight square or any straight rectangle. But what about any polygon? In the following tasks, we will look at more such special cases and go on to prove Pick's theorem for any grid polygon.

Look at the given pentagon.

a) Can you divide this pentagon into non-overlapping triangles, such that the sum of the area of all triangles is equal to the area of the pentagon? (Remember: All the vertices of each triangle should be vertices of the polygon)



How many triangles did you get?

b) Draw more polygons on your grid paper and find how many such triangles you get for each of the polygons.

We saw that any polygon can be divided into triangles. So, to prove that Pick's theorem holds for any polygon, we need to prove 2 things

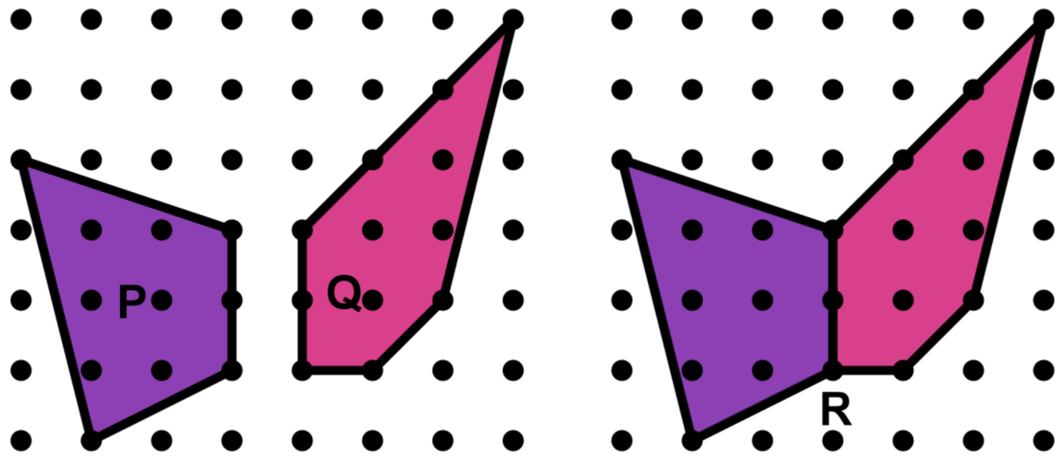
1) Pick's Theorem holds for any triangle,

2) Given two shapes for which the theorem holds, it also holds for the shape formed by joining these two shapes edge-to-edge without overlap
Then we can say that Pick's theorem holds for all polygons.

Task 8: Joining and counting!

If we put together two figures, say figure P and figure Q, in such a way that they share a boundary, and form the figure R, then we know that,

Area of (R) = Area (P) + Area (Q)



Let I_P, I_Q, and I_R be the number of grid-points in the interior of P, Q, and R respectively and B_P, B_Q, and B_R be the number of grid points in the boundary of P, Q, and R respectively.
Now, let us count I_R and B_R in terms of I_P, I_Q, B_P, and B_Q.
Let *c* be the number of grid points on the common boundary of P and Q.

Now how are the number of grid-points in the boundary of R related to those in the boundary of P and Q?

a) Can you come up with an expression for I_R and B_R in terms of I_P, I_Q, B_P, and B_Q?

$I_R = I_p + I_Q +$ ____ – ____, (Fill in the blanks) ...... (1)

$B_R = B_P + B_Q -$ ____ + ____ (Fill in the blanks) ...... (2)

(Hint: Remember the number of points of the common boundary, *c* will play an important role in this)

Now, if we assume that Pick's Theorem holds for P and Q, then what do we get?

Area (P) = Pick (P) = ________________

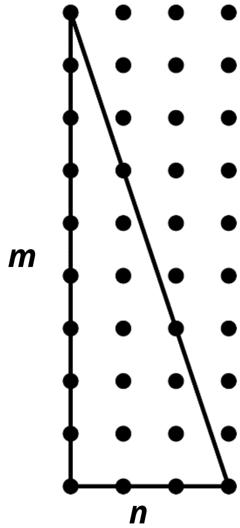
Area (Q) = Pick (Q) = ________________

Now we know that Area (R) = Area (P) + Area (Q)
So,
Area (R) = Pick (P) + Pick (Q)

(Hint: Use the expressions of I_R and B_R from (1) and (2))

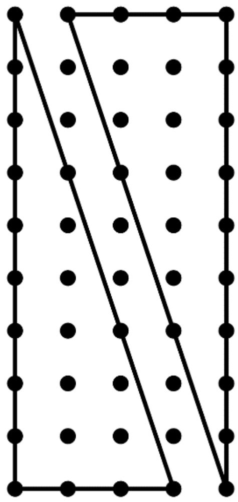
Task 9: Another special case!

In Task 7, we saw that to prove Pick's Theorem for all grid polygons we need to prove Pick's Theorem for all triangles and joining of triangles. In Task 8, we saw that Pick's theorem works for joining. So now we need to prove that Pick's Theorem holds for all triangles. But before that let us look at a very special case of triangles, namely a straight right triangle of height *m* units and base *n* units, where *m* and *n* are integers.

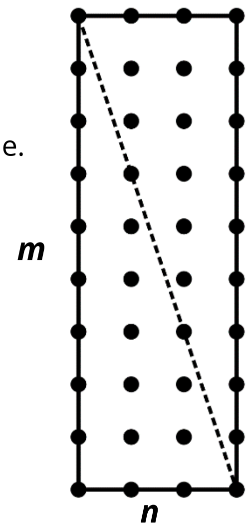


For a straight right angle of height *m* units and base *n* units,
m, and *n* are integers,

Now we can take another congruent right triangle



And join them to make a straight rectangle.



The straight rectangle we get is of length *m* units and breadth *n* units.

From Task 6 we know that,
The number of grid-points in the interior (I) of the straight rectangle = _______________
The number of grid-points on the boundary (B) of the straight rectangle = ____________

Area (straight rectangle) $= I + \frac{B}{2} - 1$ = ___________

Now, look at (1) and (2) from Task (8) where *c* is the number of points on the common boundary

$I_R = I_p + I_Q + c - 2$

$B_R = B_P + B_Q - 2c + 2$

Also, because the triangles are congruent and symmetric on the grid, we know that here,

$I_P = I_Q$

$B_P = B_Q$

(Here P and Q are the two right triangles and R is the rectangle made by joining them.

Now let

$B_{Re} =$ Number of boundary points of the rectangle, $I_{Re} =$ Number of interior points of the rectangle,

$B_{Rt} =$ Number of boundary points of the right triangle,

and, $I_{Re} =$ Number of interior points of the right triangle,

So, we get,

I_{Re} = __ × $I_{Rt} + c - 2$ ...... (3)

B_{Re} = __ × $B_{Rt} - 2c + 2$ ..... (4)

We also know that Area of rectangle = ___ × Area of right triangle

And, Pick (R) = Area (R)

So, ___ × Area(Right Triangle) = Area (R) = $I_{Re} + \frac{B_{Re}}{2} - 1$

where, $B_{Re} =$ Number of boundary points of the rectangle, and

$I_{Re} =$ Number of interior points of the rectangle,

Use (3) and (4), and check

___ × Area of (Right Triangle) = Pick (R) = ____ × Pick(P)

So, Area (Right Triangle) = Pick (Right Triangle)

Task 10

Until now we have checked that Pick's Theorem holds for straight squares, rectangles, and right triangles. We also looked at how Pick's Theorem holds even if you join two shapes edge-to-edge without overlap.

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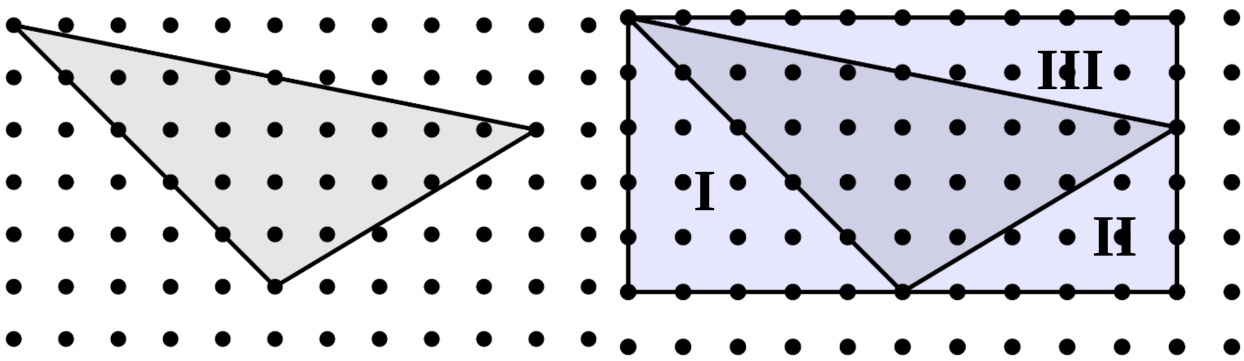
Look at the figure given below and find out what else do you need to show to prove Pick's Theorem for all polygons.



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