Exploring Non Rational Numbers

Introduction: In this learning unit, we will look at history and find out how numbers that are not rational numbers were discovered and then explore the world of these numbers by constructing them.

Part 1: Measuring using different units?

Consider two strips, Strip 1, and Strip 2 of different lengths.

Strip 1

Strip 2

Can you use the piece of string given to you and check if Strip 1 can be used to measure Strip 2? There are no markings on the string. So, the longer strip needs to be a whole number multiple of the shorter strip.

Strip ___ is ____ times Strip ___.

Let us look at one more pair of strips.

Strip 1

Strip 2

Can you use the piece of string given to you and check if Strip 1 can be used to measure Strip 2? There are no markings on the string. So, the longer strip needs to be a whole number multiple of the shorter strip.

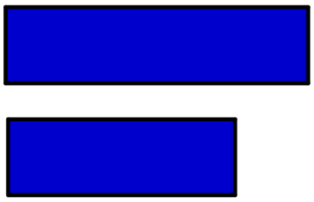
If not, let us try to measure the lengths of both strips using the "sticks" given to you. There are no markings on any of these sticks either. So, each strip needs to be a whole number multiple of the stick.

List out all the “sticks” you could use to measure both strips.

Let us do this activity with one more pair of strips.

Strip 1

Strip 2



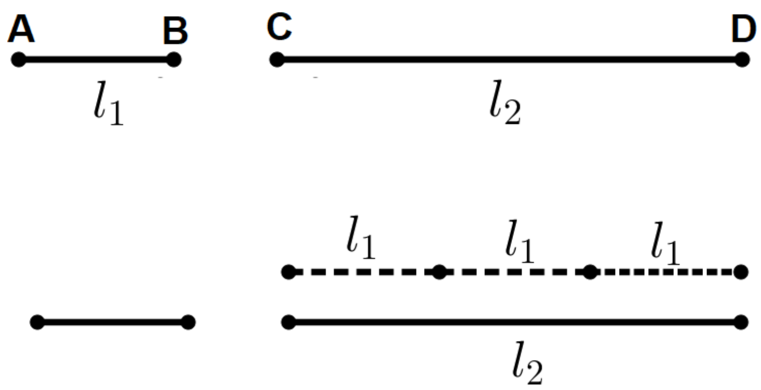
Can you use the piece of string given to you and check if Strip 1 can be used to measure Strip 2? There are no markings on the string. So, the longer strip needs to be a whole number multiple of the shorter strip.

If not, let us try to measure the lengths of both strips using the “sticks” given to you. There are no markings on any of these sticks either. So, each strip needs to be a whole number multiple of the stick.

List out all the “sticks” you could use to measure both strips.

Let us try to generalise this: Instead of the strips let us look at line segments: AB and CD of lengths l_1 and l_2. If AB fits exactly a whole number of times into CD. Then we say that l_1 is a measure of l_2, and l_2 is a multiple of l_1.

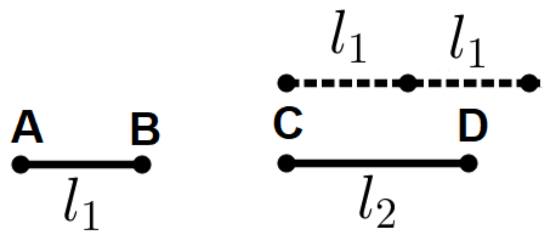
Given AB and CD, two line segments with lengths l_1 and l_2 respectively,



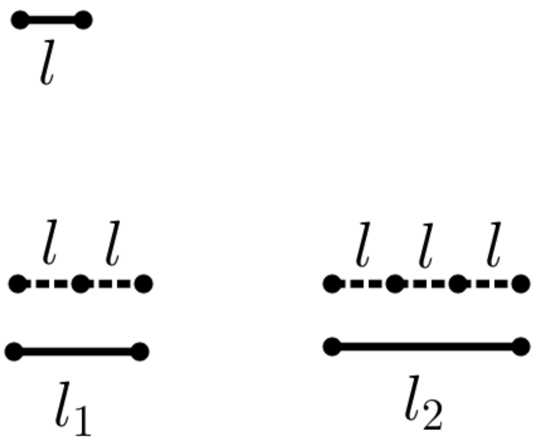
Here, we can see that, AB fits exactly 3 times into CD that means is l_2 = _____ × l_1

So, if the length of l_1 is 1 unit and the length of l_2 is _____ units.

Of course, for some other pair of line segments, it is possible that l_1 does not fit exactly a whole number of times into l_2.
For example, given two line segments AB and CD, we see that l_1 does not fit exactly a whole number of times into l_2



But, we can try to find a smaller length l, such that l fits a whole number of times in both l_1 and l_2.
For example, look at the length l given below.



We see that,

l fits into l_1 exactly ______ times and l_2 exactly _____ times. (Fill in the blanks)

Such a length l is called a common measure of lengths l_1 and l_2.

And, l_1 = ____ l and l_2 = _____ l. (Fill in the blanks)

Can you give an example of two line segments l_1 and l_2 and one common measure l of l_1 and l_2, such that $l > l_1$ and $l > l_2$?

____

Given two line segments, l_1, and l_2 such that $l_1 > l_2$ and their common measure l, what can you say about the relationship between l_1, l_2, and l?

____

Task 1

Look at the following numbers and check if one is the measure of another. If not, then try to find a common measure. Compare your answers with your friends' answers. For the remaining rows, find two pairs of l_1 and l_2, such that l_1 and l_2 are not a measure of each other.

Pair No.	l_1	l_2	**Is one a measure of the other?**	**Common measure (l)**
1	2 units	6 units	Yes	2 units, 1 unit, or 0.5 units
2	3 units	12 units		
3	5 units	3 units		
4	4 units	18 units		
5	36 units	15 units		
6			No	
7			No	

Find the greatest common measure for all the pairs of l_1 and l_2, given in the above table.
For example: For Pair 1: The greatest common measure is 2 units.
For Pair 2: The greatest common measure is _______ units.
For Pair 3: The greatest common measure is _______ units.
For Pair 4: The greatest common measure is _______ units.
For Pair 5: The greatest common measure is _______ units.
For Pair 6: The greatest common measure is _______ units.
For Pair 7: The greatest common measure is _______ units.

What can you say about the relation between the greatest common measure of two lengths and those two lengths? Discuss with your classmates.

____

Task 2

Fill in the following table

Pair No.	l_1	l_2	Common measures	Greatest common measure
1	1 unit	4.2 units		
2	3.5 units	2 units		
3	2.8 units	7 units		
4	$\frac{1}{3}$ unit	$\frac{1}{2}$ unit		
5	$\frac{1}{4}$ unit	$\frac{1}{6}$ unit		
6	$\frac{9}{4}$ units	$\frac{6}{5}$ units		

Compare your results with your friends.

Now, complete the following table by taking five pairs of two fractions (where both the numerators are not equal to 1) of your choice such that p and q are co-prime numbers and m and n are co-prime numbers.

Pair No.	$\frac{p}{q}$	$\frac{m}{n}$	Common measures	Greatest common measure
1				
2				
3				
4				
5				

Task 3: Given two line segments of lengths $\frac{p}{q}$ units and $\frac{m}{n}$ units, can you find a common measure for them? Can you find a general form for their greatest common measure?

(*p* and *q* are co-prime numbers and *m* and *n* are co-prime numbers)

Recall, all numbers of the form $\frac{p}{q}$ or $\frac{m}{n}$, where *p*, *q*, *m*, *n* are natural numbers, are called positive rational numbers.

Given two line segments if one is a measure of another or they have a common measure then these numbers (lengths of the line segments) are called *commensurable*.

So, two line segments are called commensurable if you could find a smaller line segment that could be used as a "unit" or "ruler" (or measure or a common measure) with which you can measure both the given line segments.

In Task 3, you found a common measure of any two given fractions.

So, in Task 3, you found a very important result
“Any two positive rational numbers have a common measure, which means ***All positive rational numbers are commensurable with each other!***”

This result led to an important discovery.

Now instead of the fractions $\frac{p}{q}$ and $\frac{m}{n}$, what if we look at fractions $\frac{p}{q}$ and 1, then what can you say about their common measure?

And hence you can say that any positive rational number $\frac{p}{q}$ is commensurable with ______.
(Fill in the blank)

Part 2: Any two-line segments are commensurable?

Discovery of ‘Bad’ lengths

The Pythagoras theorem is named after the famous mathematician and philosopher Pythagoras who lived in the 5th Century BC (over 2500 years ago). Pythagoras founded the Brotherhood of Pythagoreans, which was devoted to the study of mathematics.

The Pythagoreans believed that given any two line segments, one is a measure of the other or you can always find a common measure for them. That is, for any two line segments either one line segment is a measure of the other or there is a third line segment which is a common measure of both the original line segments. To put it another way, they believed that whole numbers (or counting numbers), and their ratios (rational numbers or fractions), were sufficient to describe any quantity.

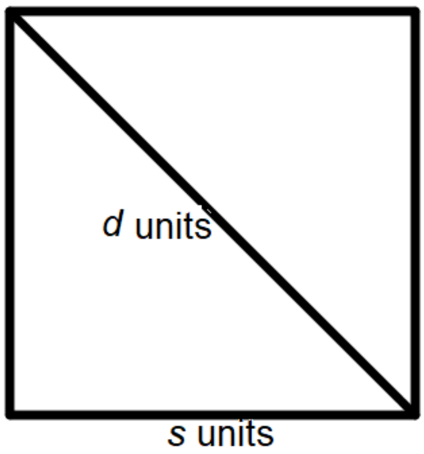
This belief was questioned when some of them found a pair of lengths that did **not** have a common measure!

Let us find out more about these two lengths.

Task 4

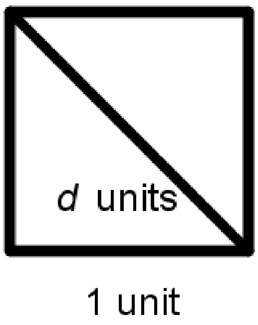
Here we have a square of side length *s* units and diagonal length *d* units.

The question was, "Do side, *s* and diagonal, *d* have a common measure?"



Now instead of any square what if we start with a square with side 1 unit?

Hint: What is the ratio of the diagonal and the side of the square? Does it change when the side changes?



So now the task is to find a common measure between two line segments, one of length 1 unit and another of length *d* units.

Pythagoras' Theorem tells us that for a right triangle with sides *a*, *b,* and *c* (*c* being the hypotenuse), we have
$c^2 = \_\_\_\_^2 + \_\_\_\_\_^2$ (Fill in the blanks)

Applying this to the right triangle inside the square we get,

$d^2 = \_\_\_\_^2 + \_\_\_\_^2 = \_\_\_\_$ (Fill in the blanks)

Let us *assume* that *d* and 1 have a common measure, let us call it *l*.

Now, there exist whole numbers *A* and *B* such that $1 = A \times l$ and $d = B \times l$.

Let us assume that *l* is the greatest common measure of 1 and *d*.

$d^2 = \_\_\_\_^2 + \_\_\_\_^2 = \_\_\_\_\_$ (Fill in the blanks)

$\Rightarrow B^- \times l^- = 2 \times A^- \times l^-$ (Fill in the blanks)

$\Rightarrow B^2 = \_\_\_\_ \times A^2$ (Fill in the blanks)

B² is an even number, because ____
Hence, *B* is an even number, because, ____
So, let us take *B* = 2*B′*
where *B′* is another whole number

Now, $d = \_\_ \times B' \times l$ (Fill in the blank)

And, $\_\_ \times B'^2 = 2 \times A^2$ (Fill in the blank)

So, $\_\_ \times B'^2 = A^2$ (Fill in the blank)

A² is an even number, because ____
Hence *A* is an even number, because, ____

Therefore, if *A* is even, then we can *A* = 2*A′* where *A′* is another whole number

Now, $1 = \_\_\_\_ \times A' \times l$ and $d = \_\_\_ \times B' \times l$
So $1 = \_\_\_ \times 2l$ and $d = \_\_\_ \times 2l$ (Fill in the blanks)

So, 2*l* is a common measure of 1 and *d*.

But, 2*l* ____ *l* (Fill in the blank with <, or, > or =)
But we choose *l* such that it was the greatest common measure!

This contradicts what we had started with!

So, our assumption that *d* and 1 are commensurable is wrong.

So, you have proved that the side of a square with side 1 unit and the diagonal of that square never have a common measure and hence are always incommensurable.

Try to modify the proof to prove that the side of any square and the diagonal of that square never have a common measure and hence are always incommensurable.

But we had proved that any positive rational number is commensurable with 1 and *d* is a positive number.

So, we see that *d* is not a rational number.
Why? ___________________________________

All such numbers which are not rational are called **irrational numbers**. In the tasks that follow we will construct some of these numbers and work with them.

Part 3: Constructing Irrational Numbers

Task 5

Let us try to geometrically construct line segments of different lengths which are irrational numbers. In the next tasks, we will geometrically construct line segments of different lengths which are irrational numbers.

1) Draw a right triangle such that two of its sides are of unit length. What can you say about the length of its hypotenuse?

2) Using the hypotenuse obtained in Task 5 as one leg and one leg with unit length, draw a right angle and complete the triangle. What is the length of the hypotenuse of the new triangle?

3) Continue this process for a minimum of 5 steps.

4) Draw a similar spiral starting with one of the sides of the triangle as 6 units and the other as 1 unit instead of both sides of 1 unit. Continue for a minimum of 5 steps. What can you say about the length of the last line segment you constructed?

5) How will you construct line segments whose length is equal to the following numbers? Give justifications as to why your answers are correct.

a) $\sqrt{32}$

b) $\sqrt{40}$

c) $\sqrt{50}$

d) $\sqrt{63}$

Task 6:

Draw some more triangles with irrational numbers as lengths.

1) Draw a right triangle such that its two perpendicular sides are $\sqrt{2}$ and$\sqrt{3}$ ($\sqrt{2}$ and $\sqrt{3}$ can be drawn using the techniques you found in the first part of this learning unit.) What is the length of its hypotenuse?

2) If you draw a right triangle such that the two right angle sides are $\sqrt{m}$ and $\sqrt{n}$, what is the length of the hypotenuse?

3) Is it possible to draw a right triangle whose all sides are integers? If yes, then draw at least two different right triangles having this property. What kind of numbers did you get as side lengths? If not, give reasons.

4) Is it possible to draw a right triangle such that the two right-angle sides are integers and the hypotenuse is an irrational number? If yes, then draw at least two different right triangles having this property. What kind of numbers did you get as side lengths? If not, give reasons.

5) Is it possible to draw a right triangle such that the hypotenuse is an integer and one of the other sides is also an integer and the third side is an irrational number? If yes, then draw at least two different right triangles having this property. What kind of numbers did you get? If not, give reasons.

6) Is it possible to draw a right triangle such that the two right-angle sides are irrational numbers and the hypotenuse is an integer? If yes, then draw at least two different right triangles having this property. What kind of numbers did you get? If not, give reasons.

7) Is it possible to draw a right triangle such that the hypotenuse is an irrational number and one of the other sides is also an irrational number and the length of the third side is an integer? If yes, then draw at least two different right triangles having this property. What kind of numbers did you get? If not, give reasons.

8) Is it possible to draw a right triangle whose all sides are irrational numbers? If yes, then draw at least two different right triangles having this property. What kind of numbers did you get? If not, give reasons.

Until now we have found a lot of irrational numbers and have also drawn them. But if you notice all of them are of the type $\sqrt{n}$, where *n* is a natural number. But there are a lot of more irrational numbers like $\sqrt[3]{n}$, or $\sqrt[4]{n}$, $\sqrt[m]{n}$ or where *n* is a natural number and *m* ≥ 2 or the famous π. Do try to find out whether you can construct some of these. For which *n*'s, can you construct $\sqrt[3]{n}$, or $\sqrt[4]{n}$ or for what *m* can you construct $\sqrt[m]{n}$, for all *n*?

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