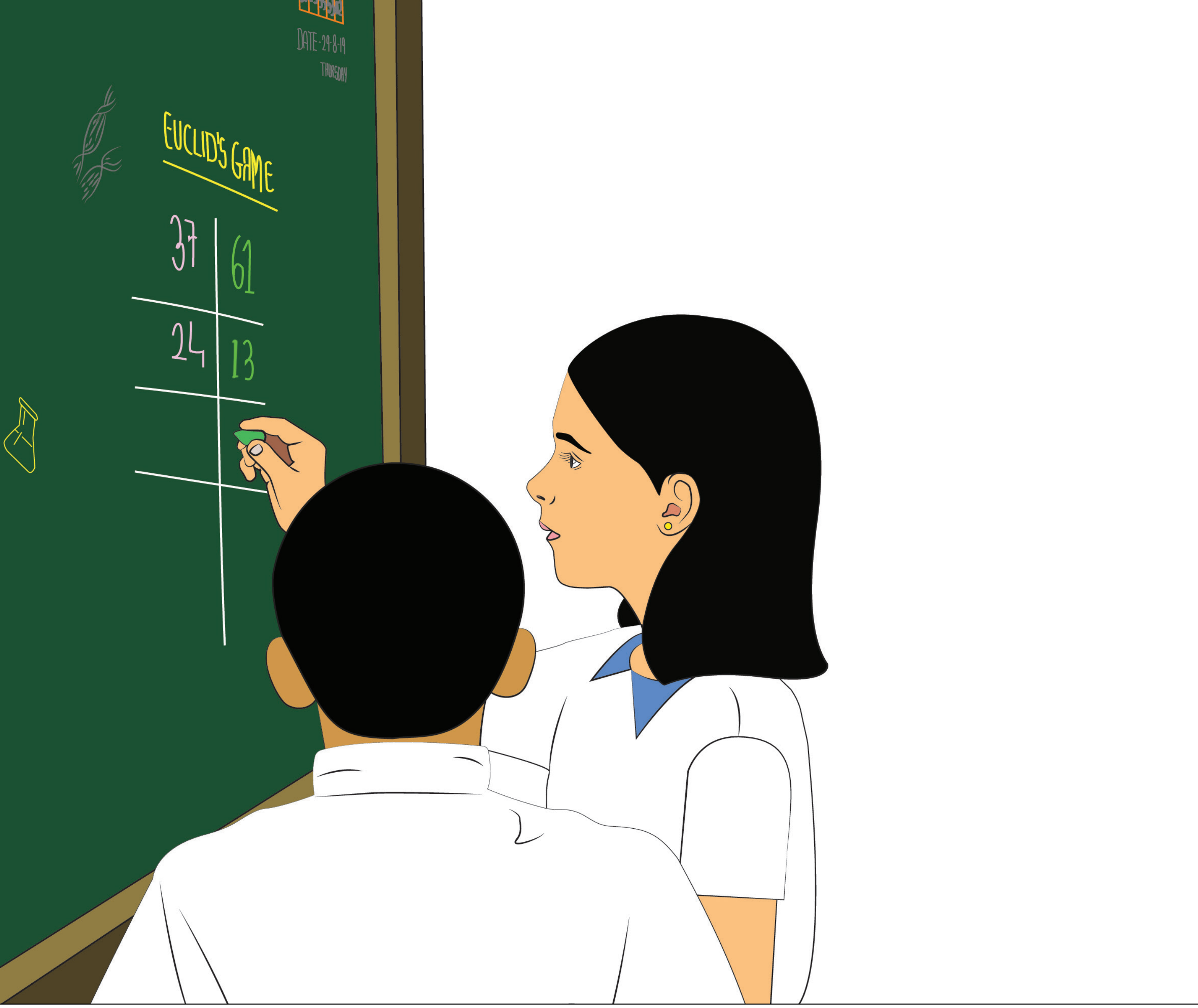

8.1

Euclid's game

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8.1. Euclid's game

Introduction

Playing games is a lot of fun. Today you are going to play a game that involves numbers, and you will find a way to win the game, always!

Task 1: Play the Euclid's game

Materials: Blackboard, chalk, sheets of paper

a. This is a two-player game.
b. The rules of the game are as follows.
   - You can decide who plays first.The first player, say Player 1, writes down a number that is between 1 and 100, including both. Let us call this number 'A'. The second player, say Player 2, can write down another number of his/her choice. Let's call this number 'B'.
   - Now, the first player will write the number (A – B) or (B – A), whichever is positive. Let's call this number 'C'.
   - Next, it is the second player's turn. He/she has a choice. He/She can either write the difference between C and A or the difference between C and B. However, if one of these differences is already in the list (i.e., if it is A or B or C) then it cannot be written again. (All differences are taken to be positive.)
   - Similarly, in subsequent turns, the players take turns to write a number which is the difference between any two numbers in the list, provided the number itself is not already present in the list.
   - The game ends when it is not possible to write any new number.
   - The person who writes the last number will be the winner.

Let us look at a sample run of the game.

- Suppose, the first player writes 12. The second player has 99 choices to choose his/her number (as the upper limit is 100).
- Suppose, the second player chooses 16, then the first player can only write 4, i.e. the difference between 16 and 12.
- The second player then writes 8, the difference between 12 and 4. Note that the player could not have written the difference between 16 and 4, as 12 is already in the list.
- Now there is no possibility of writing new numbers, so the game ends with the numbers 4, 8, 12, and 16 appearing in the list (12, 16, 4, 8 in the order of appearance).
- There are four numbers in the list, and the second player is the winner, as he/she wrote the last number 8.

Play this game with your partner multiple times. Study the lists of numbers that you got for each game and record your observations in the table below. For the last column, where you record the winner, mention whether Player 1 (who chose the first number) won or Player 2 (who chose the second number) won.

Initial Numbers		The smallest number in your sequence	The largest number in your sequence	All numbers in a sequence (in ascending order)	How many numbers are there in your sequence?	Winner
Player 1	Player 2					

Table 1

Task 2: Predict the numbers in the list

Let us assume that the following are the initial numbers in the game. Based on these, can you predict the numbers that you will arrive at, while playing the game?

(**Hint:** If you are stuck, look at the table you just made. See if there is any relationship between the initial numbers and the numbers in the list.)

a. Predict all the numbers in the list if:

i) The initial numbers are 9 and 15.

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ii) The initial numbers are 20 and 9

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iii) The initial numbers are 13 and 17.

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iv) The initial numbers are 7 and 35.

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b. How did you predict the numbers for each example? Did you notice any patterns across the examples?

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c. Now that you know the strategy for finding the list, can you predict a strategy that will ensure that one of the players will always win this game? (Which player can adopt this strategy and always win ?)

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Task 3: Looking for proofs of some conjectures

Some students made these interesting observations after playing a few rounds of the game:

Observation 1: The smallest number in the final list is the HCF of the initial pair of numbers.

Observation 2: All and only the multiples of this smallest number up to the largest number appear in the list.

a. Can you figure out why this happens for every pair of numbers?

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Let us look at the two observations.

- Observation 1 says the following:
  a) The smallest number in the list divides both the initial numbers.
  b) The smallest number is not just any common factor, but the HCF of the two initial numbers.
- Observation 2 implies the following:
  a) All the numbers in the list are multiples of the smallest number in the list,
  b) All the multiples of the smallest number up to the largest number appear in the list.

2. We need to prove or justify these observations. Can you think of the ways of doing this?

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Points to ponder

a. Do all pairs of numbers allow for a winning strategy? If not, what kinds of numbers will allow for a winning strategy?

b. What happens if you allow for the first three numbers to be random? Say, by making it a three-player game?

References

- Euclid's game: https://www.cut-the-knot.org/blue/EuclidAlg.shtml
- The optimal strategy in Euclid's game: https://math.stackexchange.com/questions/754461/optimal-strategy-in-euclids-game