Exploring Unit Circle and Trigonometric Functions' Relations with a Multipronged Educational Approach

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The mathematics textbooks and classroom discourse offer a limited platform for the students to explore the importance of unit circle in the area of trigonometry. Students often perceive triangle trigonometry, circle trigonometry, and graphs of trigonometric functions as disjoint entities, and discussions across the unit circle can help to show their interconnections. In the present study, we comment on how the unit circle is discussed in trigonometry chapters in textbooks and classroom discourse. Building on the analysis, we propose constructing some activities for students to explore the interconnections more explicitly.

Keywords: Trigonometry, Unit Circle, Triangle, Conceptual Connections

Introduction

Trigonometry is an area in school-level mathematics that deals with the relationships between angles and sides of triangles. It also serves as an important tool for explaining proofs of many laws in physics, astronomy, and engineering disciplines. Many studies in school-level trigonometry have discussed students' difficulties in trigonometry (Brown, 2006; Rosjanuardi & Jupri, 2022; Weber, 2005). To highlight a few, studies suggested students' limited understanding of sine and cosine as coordinates on the unit circle; as graphical distances; and as ratios of sides of a reference triangle. These and similar studies also indicate the importance of making the connections of unit circle with the linear distances and angles more elaborate for students. One of the approaches is the basics of trigonometry to be introduced using the unit circle trigonometric function definitions - connecting them to the ratio definitions - and then adopting the techniques of the ratio method for the solution of triangles (Kendal & Stacey, 1996; Rosjanuardi & Jupri, 2022). In the present study, we took a multipronged approach; firstly, we analysed the textbook discourse around unit circle in trigonometry chapters. We continued the studies by analysing the classroom discussions of triangle and circle trigonometry, interpretation of graphs of trigonometric functions, and conducted follow-up interviews with two teachers. Lastly, we developed a worksheet with 7 questions providing multiple opportunities for students to explore unit circle connections with triangle trigonometry and graphs of trigonometric functions and study students' responses.

Research Questions and Methodology

The study was guided by the following questions:

- How is triangle trigonometry aligned with circle trigonometry at the school level?
- What challenges do students face in the transition from acute-to-obtuse angle context?
- What challenges students face when they deal with trigonometric identity proofs?
- How do circle trigonometry connections with trigonometric graphs be discussed?

The textbook analysis involved a closer look of definitions and explanation of unit circle and its use in further concepts. We then followed observations of Grade 11, NCERT syllabus chapter titled 'trigonometric functions' (chapter 3) for 12 classroom sessions. The class had 27 students and was facilitated by an experienced mathematics teacher. The interview teachers included 2 mathematics teachers, out of which one teachers' class was observed. The school represents an urban school setting with English as a mode of instruction and is largely attended by students from families with sound socio-economic backgrounds.

Preliminary Analysis

During the classroom observations, we noted students' challenges in the transition from acute to obtuse angle trigonometry. For instance, while discussing the proof of $\cos(x+y) = \cos x \cos y - \sin x \sin y$ trigonometric identity, students agreed with the coordinates of angles 'x' as $(\cos x, \sin x)$ in the first quadrant (acute angles), but struggled with relating this notion in other quadrants. It was clear that students could not identify the coordinates of an obtuse angle 'x' as $(\cos x, \sin x)$. In the same proof, students did not connect with the key observations of needing to take the negative angle '$-y$' instead of '$-x$' even after probing on some occasions. It may sound trivial, but in a few instances; the students were hesitant to change the order of $\cos x + \cos y$ if the teacher started it as $\cos y + \cos x$. Some components of chapter 3 such as restrictions of identities and relevance of radian measure were not discussed in the class due to time constraints.

During the dialogue with teachers, one teacher mentioned not elaborating obtuse angle coordinates cases explicitly and it was mainly left for students to explore on their own. Similarly, teachers also expressed not being sure about their understanding of the need of a negative angle in the proof of identity. Furthermore, students are expected to find trigonometric ratios of 0° and 90° as special cases while discussing triangle trigonometry. However, teachers mentioned that students face challenges in accepting definitions built on approximations of trigonometric ratios when doing this exercise. Teachers also commented about the properties of triangles being eliminated from the recent curricula.

Discussions and Work in Progress

Based on the textbook analysis, classroom observations and follow-up teacher interviews, we felt that students have limited ways to appreciate unit circle and its connections with triangle trigonometry and trigonometric graphs. In classroom discourse, the connections of unit circle and trigonometric graphs are dealt as separate entities due to the textbook constraints. Further, the redefining of trigonometric ratios as coordinates of the unit circle is merely added as a one-liner without any emphasis on it and detailed explanation. Activities allowing students to explore more-detailed generalisation to the unit circle; and further to its connections with the graphs of trigonometric functions can be a useful addition. Moreover, it is important for students to first understand why such a generalisation, particularly of unit circle, is needed. Some key thoughts guiding the development of these activities spurred from the above observations, which are: a) the exercise of finding trigonometric ratios of 0° and 90° as special cases can be incorporated with the unit circle rather than dealing it in the triangle trigonometry. b) the properties of triangles (including sine and cosine rules) are being used in NCERT XI physics classes but their derivations being removed from previous mathematics courses may leave a gap in building appropriate connections, so a careful consideration for re-inclusion of properties of triangles is needed. With the cosine rule, students can easily derive the trigonometric identities.

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