Counting Areas

Introduction: In this unit we will be finding a new way to get areas of polygons. Here we will not be calculating or approximating areas of polygons but figuring newer ways to find areas of polygons.

Let us start with a story.

King Bahubali and his elephants

King Bahubali loved elephants so much that he kept a herd of them. In fact, he planted his coconut garden in such a way that it looked like an elephant when viewed from his terrace!
But the elephants would walk around the garden and destroy it. So, the king put a fence around the garden to keep the elephants away as seen in Figure 1. The trees were planted on a square grid, with one tree at each grid point; the distance between any two consecutive grid points was 1 unit, to provide sufficient space for each tree. Can you find the area of the coconut garden?
If you cannot solve it now, go ahead with the remaining tasks, and you will be able to do this at the end of the tasks!!

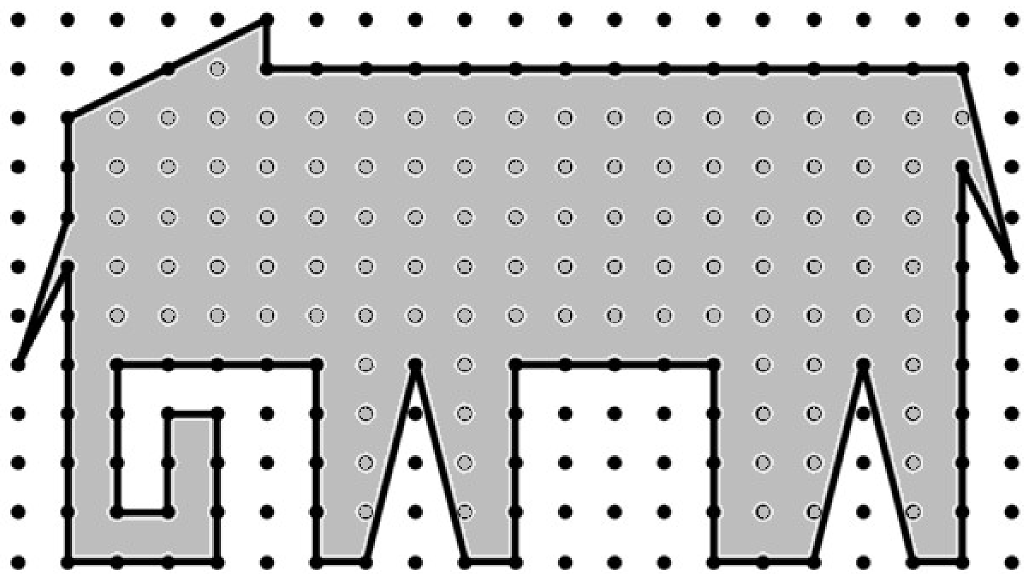


Figure 1

Task 1: Given here are some shapes. Find the area of each and complete the given table.

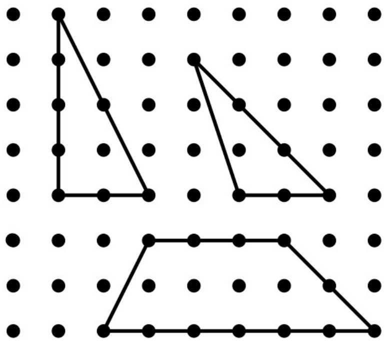


Figure 2

Shape	Area in Sq Units
(i)	
(ii)	
(iii)	

Table 1

Task 2:

Given below are some shapes. Find the area of each and complete the given table.

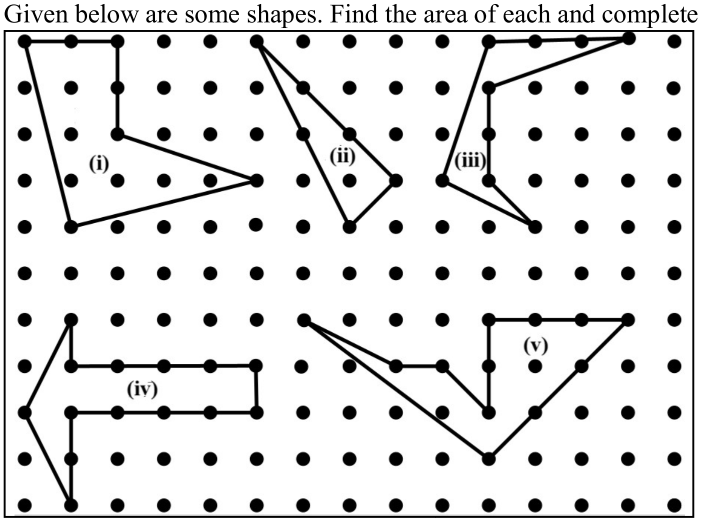


Figure 3

Shape	Area in Sq Units
(i)	
(ii)	
(iii)	
(iv)	
(v)	

Table 2

Task 3: Some more shapes!

a) Find the area of the following shapes. Also count the number of grid-points on the boundary of each shape in the Figure 4, and fill Table 2.

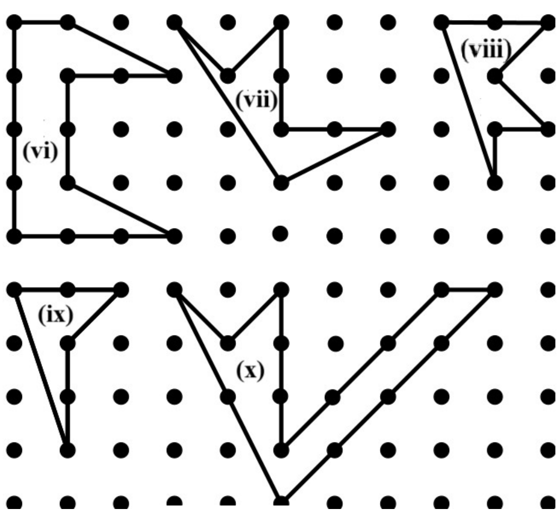


Figure 4

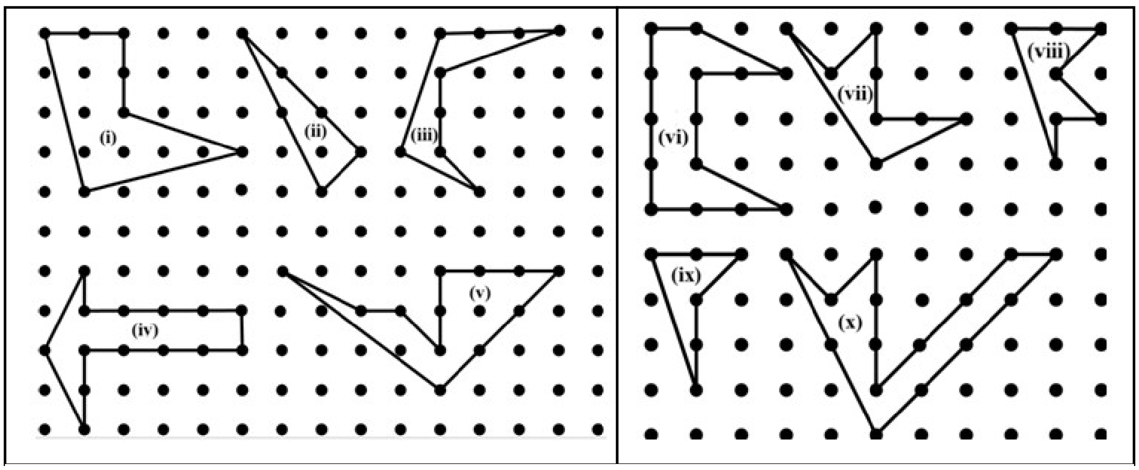
Shapes	Area in Sq Units (A)	Number of grid-points on the boundary (B)
(vi)		
(vii)		
(viii)		
(ix)		
(x)		

Table 3

b) Do you see any relation between the area of a shape and the number of grid-points on its boundary?

c) Does the same relation hold for shapes (i) to (v) in Task 1? If not, for which ones does the relation hold?

Let us look at shapes from Task 1 and Task 2 together and compare.



Shapes from Task 2 Shapes from Task 3

Figure 5

Shape	Area in Sq Units	Number of grid-points on the boundary (BP)	
(i)			
(ii)			
(iii)			
(iv)			
(v)			

Table 4

(A) The relation holds for Shapes numbers _______________.

(B) The relation does not hold for Shapes numbers _______________.

Task 4: Finding the expression!

a) In Task 3c, how do the shapes in (A) and (B) differ?

b) How would you modify the relation in Task 3b such that it holds for all shapes?

Task 5: Making some more shapes

Draw five more shapes on the grid provided below and check if the relation holds for these shapes as well.

(a) Are you sure that it will hold for any shape that you may draw on the grid?

(b) What are the properties common to the shapes for which this relation holds?

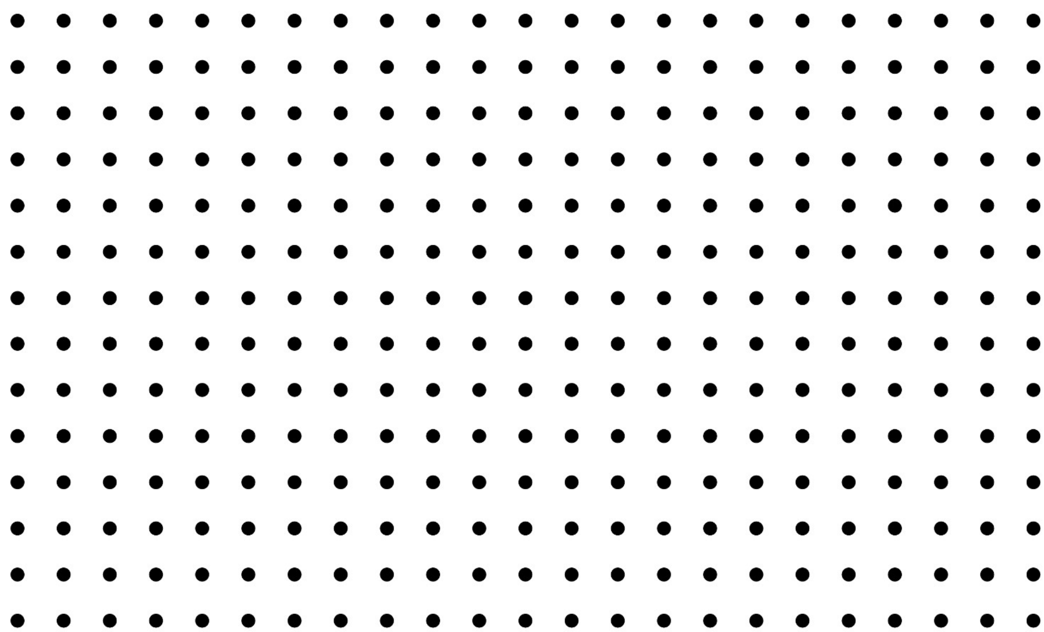


Figure 6

We have looked at some polygons and found an expression for their area by just counting the boundary and the interior points.

For any shape S, with I = number of grid-points in its interior and B = number of grid-points on its boundary. Let is define Pick(S) as:

$$\textbf{Pick(S)} = I + \frac{B}{2} - 1$$

From the polygons that we have drawn, we saw that for a shape S,

Pick(S) = Area(S)

This relation is called Pick's Theorem.

Now let us look at some special polygons and prove that this theorem holds for them too.

Task 6: Special cases!

In this part, we will look at some special cases. We will find some special quadrilaterals like straight squares and straight rectangles. Do you know what is a straight rectangle and square?

Let us find out.

Look at the rectangles in Figure 7, we will call Rectangles 2, 4, and 6 as straight rectangles and Rectangles 1, 3 and 5 as slanted rectangles. Note that Rectangle 6 is also a straight square and Rectangle 1 is a slanted square.

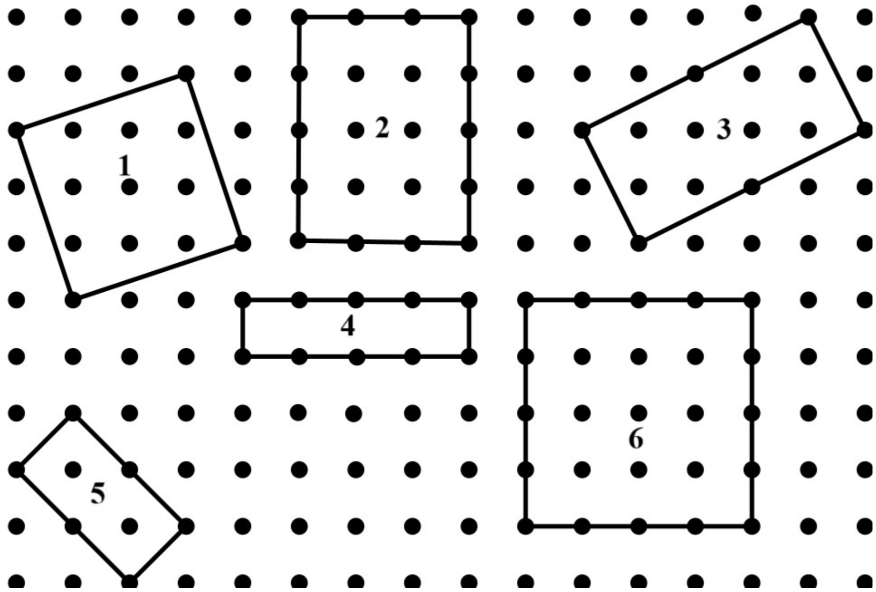


Figure 7

Let us try to prove Pick's Theorem for a straight square and a straight rectangle. What do we mean by proving Pick's Theorem for a straight square and rectangle?

We know that for a shape, S: Pick(S) = $I + \frac{B}{2} - 1$ ... Pick's Theorem

So to prove Pick's Theorem, we have to count the interior points and boundary points of a straight square or a straight rectangle. Then put them in the Pick's Formula and see if it is equal to the area of a straight square and a straight rectangle which we know.

We know that for a square of side m units, the area of a square is m^2

And we know that for a rectangle of sides m and n units, the area of the rectangle is mn.

So we have to show that,

Pick(Straight Square) = m^2 and

Pick(Straight Rectangle) = mn

We want to prove Pick's Theorem for a straight square.

But before that let us look some specific straight squares. These might help us in proving Pick's Theorem for a straight square.

i. A square of side 6 units.

On a line segment of 6 units, the number of points on the line segment = (6 + 1).

Total number of points of the square = (____)^2 = _____

Notice that total number of points are boundary points and interior points counted together.

We have four line segments of length 6 units as the boundary of the square.

So, the number of points on the boundary

B = __________ = 4 × ____

(Remember to check for points which have been counted twice, namely the ones at the corners)

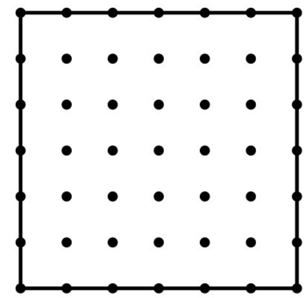


Figure 8

Also, total number of points of the square

= Number of points inside the square + Number of points on its boundary

Number of points in the interior of the square,

I = Total number of points – Points on the boundary

I = ___________ $-$ ___ = ____

So, Pick(Square) = $I + \frac{B}{2} - 1$ = ____________.

We know that, Area(Square) = ____

So, Area(Straight Square) _____ Pick(Straight Square) (Fill in the blanks with <, > or = sign)

ii. A square of side 5 units.

On a line segment of 5 units, the number of points on the line segment = (5 + 1).

Total number of points of the square

= Number of points inside the square + Number of points on its boundary

= (____)^2 = _____

Notice that the total numbers of points are boundary points and interior points together.

We have four line segments of length 5 units as the boundary of the square.

So, the number of points on the boundary

B = __________ = 4 × ____

(Remember to check for points which have been counted twice, namely the ones at the corners)

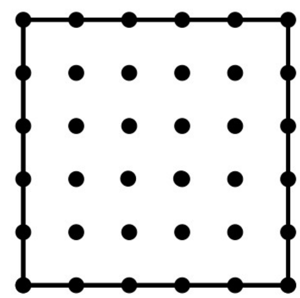


Figure 9

Number of points in the interior of the square, I =

Total number of points – Points on the boundary

I = _________________ $-$ __________ = _________

So, Pick(Straight Square) = $I + \frac{B}{2} - 1$ = ____________.

We know that, Area(Straight Square) = _____

So, Area(Straight Square) _____ Pick(Straight Square) (Fill in the blanks with <, > or = sign)

We saw two examples of squares where Pick's Theorem is true. But we have not proved that it is always true for any straight square. But we might be able to use these examples to prove Pick's Theorem for any straight square.

Let us look at a general $m \times m$ square.

For a straight square of side m units

Notice: On a line segment of *m* units, the number of grid points on the line segment is (m + 1).

Total number of points of the square = $(m + 1)^2$ = ______________

Also, total number of points of the square
= Number of points inside the square + Number of points on its boundary
Notice that total points are boundary points and interior points counted together.

We have four such line segments as the boundary of the square.
So, the number of points on the boundary
B = ______________ = 4 × _____

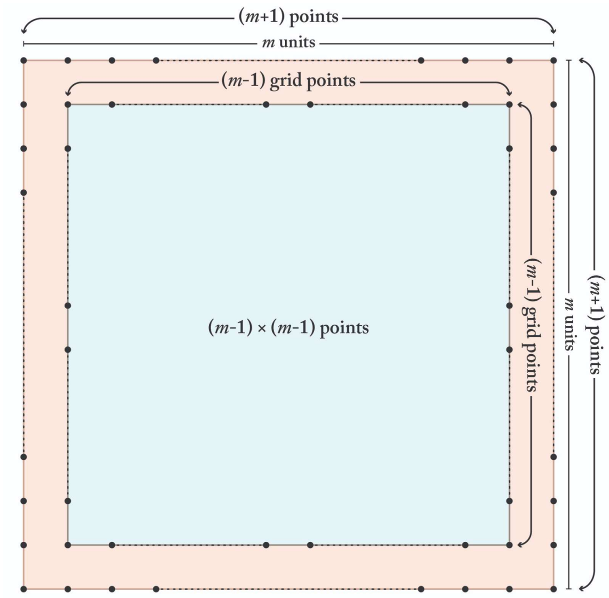


Figure 10

(Remember to check for points which have been counted twice, namely the ones at the corners)

Number of points in the interior of the square = I and
I = Total number of points – Number of points on the boundary
So, I = _____________________ − ______________ = _____________
So, Pick (Straight Square) = $I + \frac{B}{2} - 1$ = _____________.
We know that, Area (Straight Square) = ____
So, Area (Straight Square) _____ Pick (Straight Square) (Fill in the blanks with <, > or = sign)

a) For a straight rectangle of length *m* units and breadth *n* units

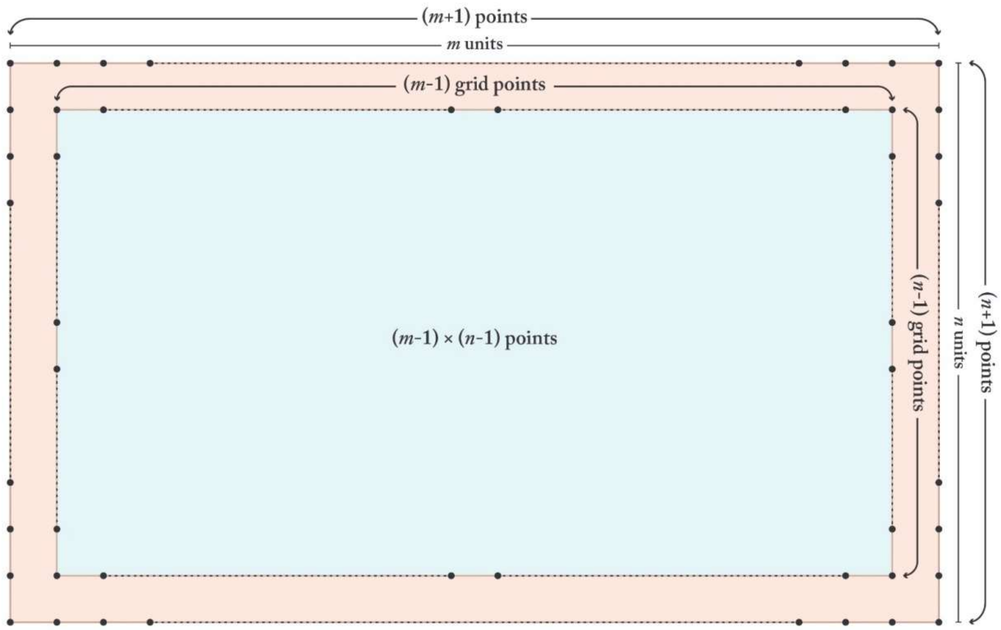


Figure 11

On a line segment of *m* units, the number of points on the line segment is $(m + 1)$.
And, on a line segment of *n* units, the number of points on the line segment is $(n + 1)$.
Total number of points of the rectangle = $(m + 1)(n + 1)$ = _______________
Also, total number of points of the rectangle = Number of points inside the rectangle + Number of points on its boundary
On the boundary, there are two line segments of *m* units and two line segments of *n* units.
So, the number of points on the boundary, B = ___________
(Remember to check for points which have been counted twice, namely the ones at the corners)
Number of points in the interior of the straight rectangle, I
= Total number of points – Points on the boundary = _____ − _____
The number of points in the interior, I = _________
Pick (Rectangle) = $I + \frac{B}{2} - 1$ =_____
We know that, Area (Straight Rectangle) =_______
So, Area (Straight Rectangle) _____ Pick (Straight Rectangle) (Fill in the blanks with <, > or = sign)

So, for a straight rectangle and a straight square, P, we proved that
Pick (P) ___________ **Area (P)** (Fill in the blanks with <, >, or = signs)

This relation is called Pick's Theorem.

So, we have proved Pick's theorem for a very special class of shapes namely straight squares and straight rectangles. We still need to prove Pick's Theorem for any polygon.

Task 7: What about any polygon?

We have proved that Pick's theorem holds for any straight square or any straight rectangle. But can we say that Pick's theorem holds for any polygon?

In the following tasks we will look at more such special cases and go on to prove the Pick's theorem for any grid polygon.

Look at the given pentagon.

a) Can you divide this pentagon into non-overlapping triangles, such that sum of area of all triangles is equal to the area of the pentagon?

(Remember: All the vertices of each triangle should be vertices of the polygon)

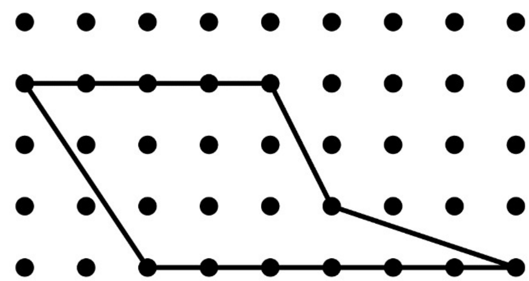
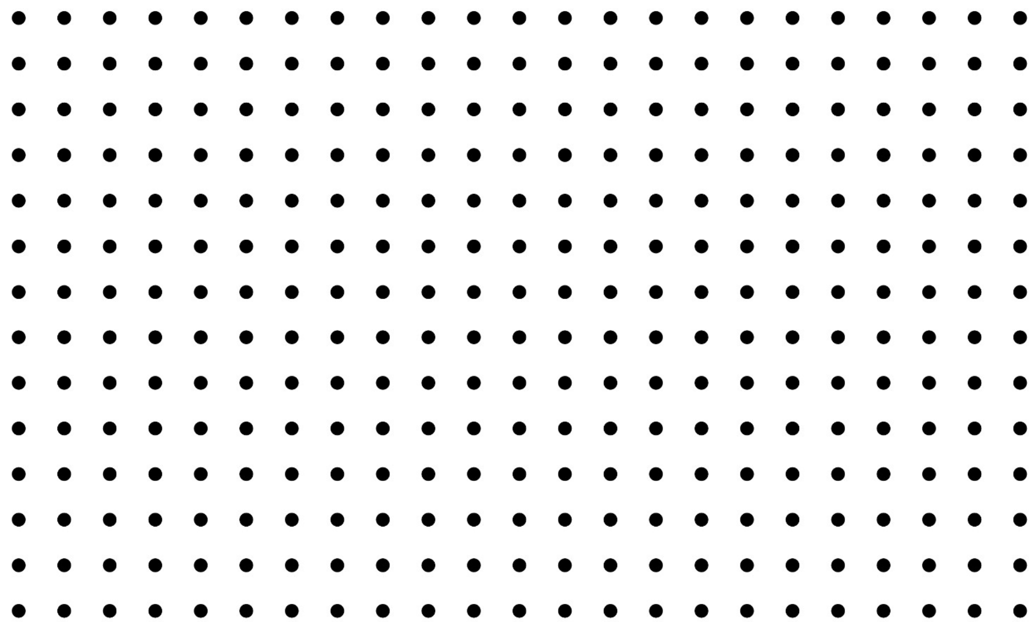


Figure 12

How many triangles did you get?

b) Draw more polygons on the grid paper given to you and find how many such triangles did you get in each of the polygons?



We saw that any polygon can be divided into triangles. So in order to prove that Pick's theorem holds for any polygon, we need to prove 2 things

1) Pick's Theorem holds for any triangle,

2) Given two shapes for which the theorem holds, it also holds for the shape formed by joining these two shapes edge-to-edge without overlap

Then we can say that Pick's theorem holds for all polygons.

Task 8: Joining and counting!

If we put together two shapes, say Shape P and Shape Q, in such a way that they share a boundary, to form Shape R then we know that,

Area of (R) = Area (P) + Area (Q)

Look at the shapes P and Q in Figure 13. Imagine P and Q are joined together to get the shape R.

Let us assume that Pick's Theorem holds for shapes P and Q. So we know

Pick P = $(I_{\mathrm{P}} + \frac{B_{\mathrm{P}}}{2} - 1)$ = Area (P) and Pick Q = $(I_{\mathrm{Q}} + \frac{B_{\mathrm{Q}}}{2} - 1)$ = Area (Q)

We have to prove that: Pick R = $(I_{\mathrm{R}} + \frac{B_{\mathrm{R}}}{2} - 1)$ = Area (R)

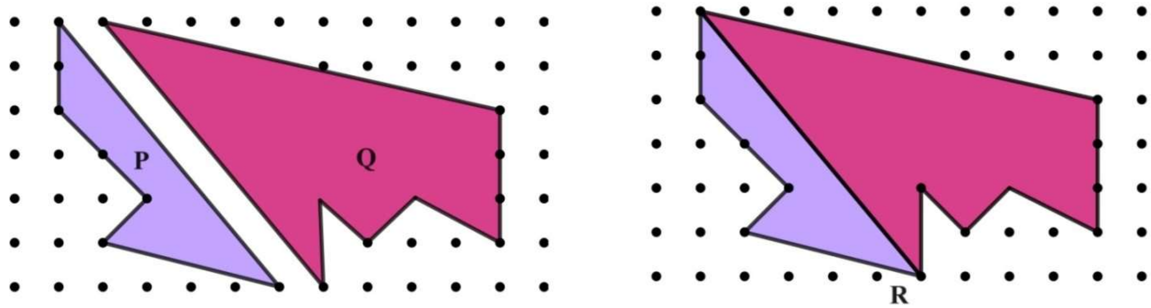


Figure 13

Let I_{P}, I_{Q} and I_{R} be the number of grid-points in the interior of P, Q and R respectively and B_{P}, B_{Q} and B_{R} be the number of grid points in the boundary of P, Q and R respectively.

Now, let us count I_{R} and B_{R} in terms of I_{P}, I_{Q}, B_{P} and B_{Q}.

Let c be the number of grid points on the common boundary of P and Q.

(a) What is the relation between the numbers of boundary points of P, Q related to the number of boundary points of R?

(b) Can you come up with an expression for I_{R} and B_{R} in terms of I_{P}, I_{Q}, B_{P} and B_{Q}?

$I_{\mathrm{R}} = I_{\mathrm{P}} + I_{\mathrm{Q}} +$ ____ $-$ ____, (Fill in the blanks) …… (1)

$B_{\mathrm{R}} = B_{\mathrm{P}} + B_{\mathrm{Q}} -$ ____ $+$ ____ (Fill in the blanks) …… (2)

(Hint: Remember the number of points of the common boundary, c will play an important role in this)

Now, if we assume that Pick's Theorem holds for P and Q, then what do we get?

Area (P) = Pick (P) = __________________

Area (Q) = Pick (Q) = __________________

Now we know that Area(R) = Area (P) + Area (Q)

So,

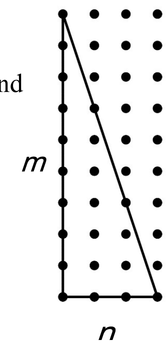
Area (R) = Pick (P) + Pick (Q)

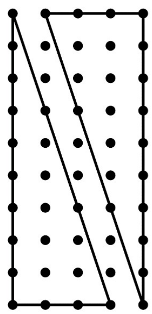
(Hint: Use the expressions of I_{R} and B_{R} from (1) and (2))

Task 9: Another special case!

In Task 7, we saw that to prove Pick's Theorem for all grid polygons we need to prove Pick's Theorem for all triangles and joining of triangles. In Task 8, we saw that Pick's theorem works for joining. So now we need to prove that Pick's Theorem holds for all triangles. But before that, let us look at a very special case of triangles, namely a straight right-angle triangle of height *m* units and base *n* units, (*m* and *n* integers).

Take a straight right-angle triangle of height *m* units and base *n* units (where *m* and *n* are both integers).





Now we take another congruent right-angle triangle.

And join them to make a straight rectangle.

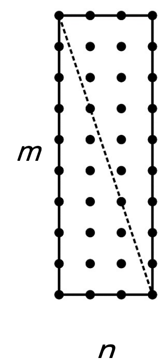


Figure 14

The straight rectangle we get is of length *m* units and breadth *n* units.

From Task 6, we know that,

The number of grid-points in the interior (I) of the straight rectangle = _______________

The number of grid-points on the boundary (B) of the straight rectangle = ____________

Area (Straight rectangle) $= I + \frac{B}{2} - 1$ = ___________

Now, look at (1) and (2) from Task (8) where c is the number of points of the boundary

$I_P + I_Q = I_R$ − __ + ___

$B_P + B_Q = B_R$ + ___ − ___

This is because the triangles are congruent and symmetric on the grid. So the interior points and the boundary points also would be the same in both these triangles. Hence, we can say that, $I_P = I_Q$ and $B_P = B_Q$

(Here P and Q are the two right-angle triangles and R is the rectangle made by joining them.)

So, we get,

___$\times I_P = I_R - c + 2$

___$\times B_P = B_R + 2c - 2$

We also know that, Area of rectangle = ___ × Area of right-angle triangle

And, Pick (straight rectangle) = Area (straight rectangle)

So, ___ × Area (Right-Angle Triangle) = Area (Re) = $I_{Re} + \frac{B_{Re}}{2} - 1$

where, B_{Re} = Number of boundary points of the straight rectangle and

I_{Re} = Number of interior points of the straight rectangle

Using the equations given above,

___ × Area of (Right-angle triangle) = Pick (R) = ____ × Pick(P)

So, Area (Right-angle triangle) = Pick (Right-angle triangle)

Task 10: Triangle inside a rectangle?

Till now we have checked that Pick's Theorem holds for straight squares, straight rectangles, and straight right-angle triangles. We also looked at how Pick's Theorem holds even if you join two shapes edge-to-edge without overlap.

Look at Figure 15 given below and find out what else do you need to show to prove Pick's Theorem for all polygons.

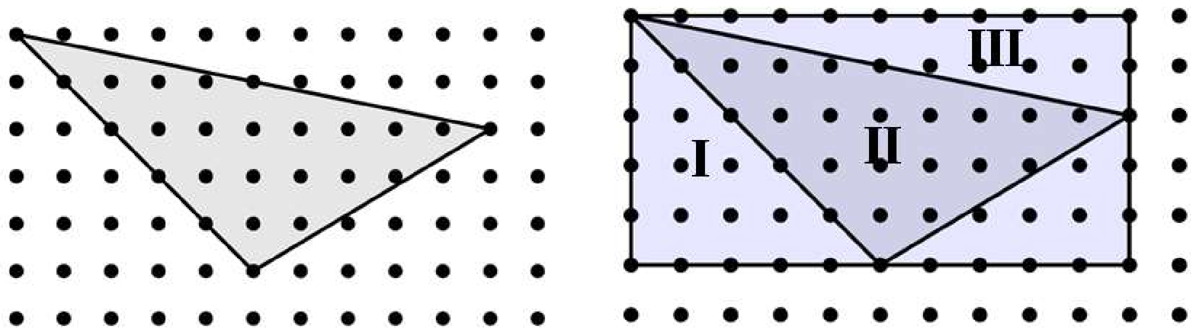


Figure 15

Story so far:

In this Learning Unit, we started with trying to find a relation between the area of a grid polygon and its boundary and interior points.

Once we got the relation, we set out to prove it. For proving the relation for a general grid polygon, we started by proving the relation for any straight rectangle and a straight square. Then we noticed that any grid polygon can be divided into non-overlapping triangles such that sum of areas of these triangles is equal to the area of the polygon.

Now our goal became proving the relation of triangle, But there was a small issue. When you join two shapes to make another one, some boundary points get counted twice and some others become interior points. So we had to show that in case of joining two shapes where the relation holds, the relation holds for the larger shape too.

Now all we need to do was to prove that relation holds for any triangle. Figure 15 tell us how we can prove it.

Once we prove it, we can ready to find the area of the coconut grove.

Task 11: Find the area of Bahubali's coconut garden, using Pick's Theorem.

References

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http://www.geometer.org/mathcircles/pick.pdf
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