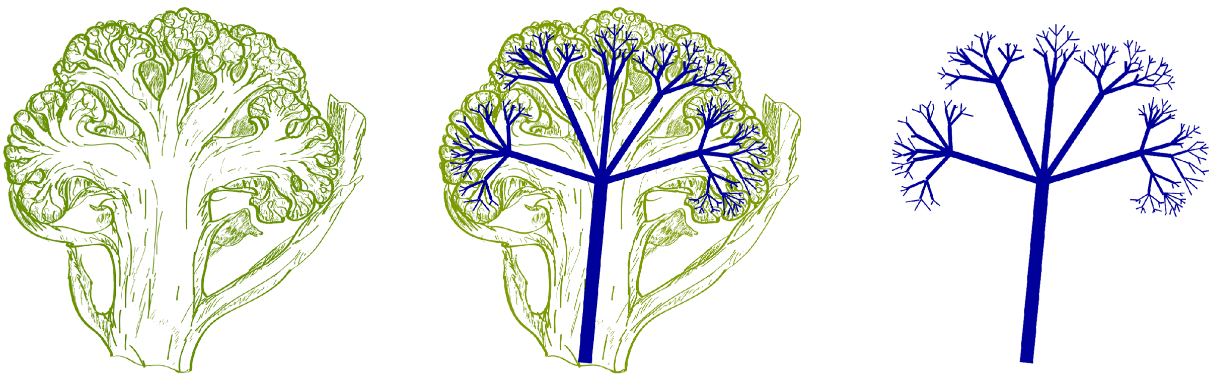
An Exploration of Fractals

An introduction to the activity: What is a Fractal?

Have you ever seen a cauliflower closely? If you observe it carefully, you will see that the smaller portions of it resemble the whole cauliflower. If you take a still smaller portion and magnify it, you will find that this too resembles the whole object. This is an example of a pattern which repeats itself, where each smaller portion is a scaled down copy of the larger portions. This leads us to the idea of self – similarity.



In fact, we are now in a position to define a fractal. Fractals are self-similar objects. By self – similar we mean that smaller parts of the object are exact scaled down copies or replicas of the whole object. Nature is full of fractals. Brocolli, certain fern leaves, clouds etc manifest self-similarity and can be categorized as fractals.

In this activity, we are going to construct and explore Fractals through simple geometric constructions.

Task 1: Sierpinski triangle

In this activity we shall construct the Sierpinski Triangle which has been named after the mathematician Waclaw Sierpinski.

A fractal can be generated in stages by means of an iterative process. The first stage is referred to as stage 0 where we start with a geometrical figure. To obtain stage 1 a geometric construction is performed on the figure at stage 0. To obtain stage 2 the construction is repeated on all parts of stage 1 and so on.

Constructing the Sierspinski Triangle.

Step 1:

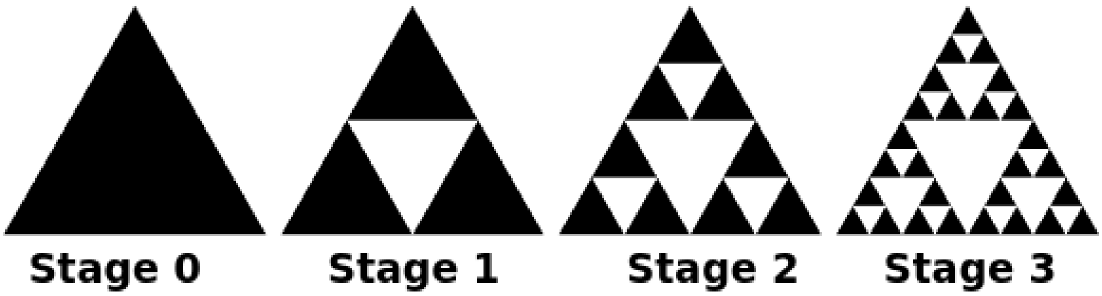
Draw an equilateral triangle. Shade or color the entire triangle. This will be called stage 0.

Step 2:

Draw another equilateral triangle of the same size as Step 1. Mark the midpoints of the three sides and join them to obtain four smaller equilateral triangles. Shade or color the three side triangles, that is, the triangles pointing upwards, leaving out the middle triangle. This figure is stage 1.

Step 3:

Construct the same figure as in stage 1 except for the shading. In each of the three triangles which were shaded in stage 1, repeat the process of joining the midpoints of the sides and shading only the triangles pointing upwards, each time, leaving out the middle triangle. This shaded figure is stage 2. The figure given below shows the Sierpinski triangle up to stage 3.



Number Patterns in the Sierpinski Triangle

Investigation 1

Count the number of shaded triangles up to stage 4 of the Sierpinski triangle. What will be the number of shaded triangles at stage 5? Complete the given table.

Stage	0	1	2	3	4	5
Number of shaded triangles	1					

Did you notice a pattern?

Stage	0	1	2	3	4	5	…………	*n*
Number of shaded triangles	1						………….	

Can you express the number of shaded triangles at any stage, in terms of the number of triangles in the previous stage?

What will happen to the number of triangles as *n* increases?

Investigation 2:

How many copies of stages 0,1 and 2 do you expect to find in stage 3 of the Sierpinski Triangle? In general how many copies of the previous stages will you find in stage *n*?

Exploring the concept of area within the Sierpinski triangle

Investigation 3

Let the area of the shaded triangle at stage 0 be *A* sq. unit, write down the shaded areas up to stage 4. What will be the area at stage 5?

Stage	0	1	2	3	4	5
Area of shaded triangles (in sq. units)	*A*					

What constant multiplier can be used to go from one stage to the next? Express the area of shaded triangles as are powers of a given fraction.

Stage	0	1	2	3	4	5	....	*n*
Area of shaded triangles (in sq. units)	*A*						....	

What happens to the area of the shaded triangles as *n* increases indefinitely?
Can you express the shaded area at any given stage in terms of the shaded area of the previous stage?

Exploring the concept of the length of the boundary within the Sierpinski triangle

Investigation 4

Suppose the side of the triangle at stage 0 is 1 unit. What is the length of its boundary? Find the lengths of the boundary at every stage 5?

Stage	0	1	2	3	4	5
Length of the boundary	3					

What constant multiplier can be used to go from one stage to the next?

Stage	0	1	2	3	4	5	…………	*n*
Length of the boundary	3						…………	

What happens to the length of the boundary of the shaded triangles as *n* increases?

Can you express the length of the boundary of the shaded triangles at any given stage in terms of the length of the boundary of the shaded triangles of the previous stage?

A spreadsheet exploration

We shall now explore the Sierpinski Triangle construction process numerically using a spreadsheet. In Column A we shall represent the stage numbers *n*. In columns B, C and D, we shall represent the number of shaded triangles, the shaded areas at various stages and the length of the boundary at various stages respectively. The steps are as follows:

Step 1:

In column A, enter 0 in cell A2 and enter the formula = A2 + 1 in cell A3. Drag the cell A3 (till A22) to generate the stage numbers till 20. This column will represent the stage numbers *n*.

Step 2:

In column B, we enter 1 in cell B2 and the formula = B2 × 3 in cell B3. Double click the cursor in the corner of the cell B3 to generate the sequence of number of shaded triangles.

Step 3:

In column C, enter 1 in cell C2 and the formula = C2 × ($\frac{3}{4}$) in cell C3. Double clicking the cell C3 will generate the sequence of shaded areas.

Step 4:

In column D, enter 3 in cell D2 and the formula = D2 × ($\frac{3}{2}$) in cell D3. Double clicking the cell D3 will generate the sequence of the length of the boundary at every stage.

Investigation 5:

Generate the data on a spread sheet and observe what happens to the column values as the stages increase. Use the Graphing feature to generate the graphs for the number of triangles, shaded area and length of the boundary at each stage. What do you observe from the graphs?

References :

Book: Hands-on-mathematics, class 7 by Jonaki B Ghosh (Author), Haneet Gandhi (Author), Tandeep Kaur (Author) .
Images: https://commons.wikimedia.org/wiki/Fractal
https://www.jstor.org/stable/10.5951/mathteacher.109.9.0693?seq=1#page_scan_tab_contents

