An Exploration of Fractals

Overview:

The purpose of this unit is to draw students' attention to one of the ways mathematics shows up in nature namely fractals. In this unit, the students will construct the Sierpinski triangle which is a fractal. They will also generalize different patterns that come up during this constructions and give arguments towards their generalizations. In the last part of the unit, the students will explore these patterns on a spread-sheet.

Minimum time required:

2 sessions of 40 min

Type of LU:

Computer lab or a classroom with a projector

Link to the curriculum:

NCERT Class 6	Chapter 13 - Symmetry
	Chapter 14 – Practical Geometry
NCERT Class 7	Chapter 10 – Practical Geometry
	Chapter 11 – Perimeter and Area
	Chapter 12 – Algebraic expressions
	Chapter 13 – Exponents and Powers
	Chapter 14 - Symmetry
NCERT Class 8	Chapter 9 – Algebraic expressions and Identities
	Chapter 12 – Exponents and Powers
	Chapter 15 – Introduction to graphs
NCERT Class 9	Chapter 3 – Coordinate Geometry

Learning objectives:

To introduce the student to the concept of fractals and self-similarity (through objects occurring in nature).

To construct the geometrical fractal Sierpinski triangle. To explore number patterns arising from the fractal constructions.

To explore the concept of area and the length of boundary through the fractal constructions.

To generalize the number patterns by finding recursive and explicit formulas.

To explore the fractal patterns numerically and graphically.

Prerequisites:

Concept of perimeter and area

Constructing an equilateral triangle

Constructing a midpoint of a line segment

Materials required:

Paper, Pencil, Magnifying Glass, Compass, Scale, Protractor, any spreadsheet program (for numerical and graphical explorations)

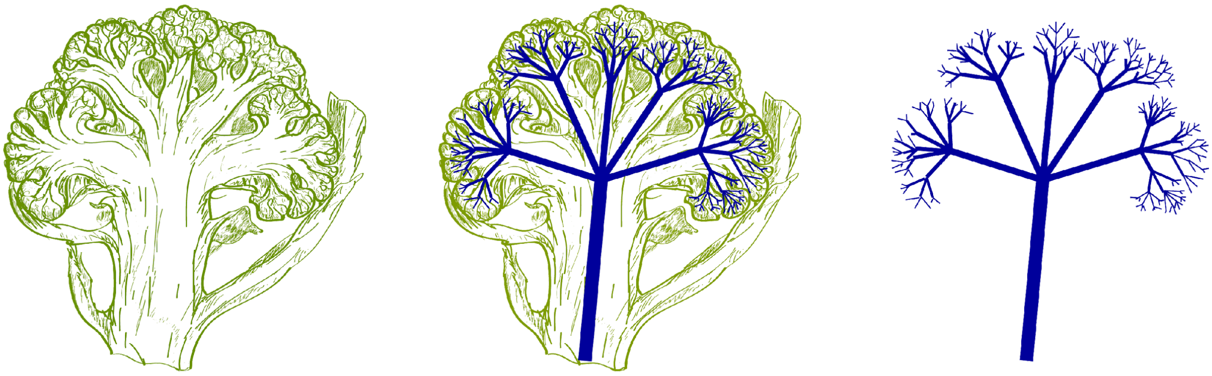
An introduction to the lesson: What is a Fractal?

If it is possible – carry a cauliflower and make some pieces of it (very small, medium and large). Now give it to your students and ask them to observe. You can take collective responses about their observations on board. You can also give the students a magnifying glass and ask them to look at the cauliflower through the glass. This might underline the self-symmetry more.

Have you ever seen a cauliflower closely? If you observe it carefully, you will see that the smaller portions of it resemble the whole cauliflower. If you take a still smaller portion and magnify it, you will find that this too resembles the whole object. This is an example of a pattern which repeats itself, where each smaller portion is a scaled down copy of the larger portions. This leads us to the idea of self – similarity.

Also, ask them what will happen if you see that very small part of cauliflower under a microscope? They might say that it will look like the bigger piece of cauliflower. If they don't you can ask will the smaller part look like a bigger part? Once they agree on this, ask them which are the other objects they have seen in nature have the same property?

Now, you can introduce the term self-similar by saying that all the objects where the smaller part resembles the bigger part are known as self-similar.



In fact, we are now in a position to define a fractal. Fractals are self-similar objects. By self – similar we mean that smaller parts of the object are exact scaled down copies or replicas of the whole object. Nature is full of fractals. Broccoli, certain fern leaves, clouds etc manifest self-similarity and can be categorized as fractals.

You can show these pictures on presentation slides as well. If you don't want to take a print out of it.

In this lesson we are going to construct and explore Fractals through simple geometric constructions.

Task 1: Sierpinski triangle

In this activity we shall construct the Sierpinski Triangle which has been named after the mathematician Waclaw Sierpinski.

A fractal can be generated in stages by means of an iterative process. The first stage is referred to as stage 0 where we start with a geometrical figure. To obtain stage 1 a geometric construction is performed on the figure at stage 0. To obtain stage 2 the construction is repeated on all parts of stage 1 and so on.

Before starting the Task 1, Ask students, what do they understand by area of a shape? And the concept of length of the boundary of a shape? Also ask them what they understand by polygon? What is the difference between regular polygon and polygon?

You can also ask which is the regular polygon with the smallest number of vertices? Do they know how to construct it?

Shown below are 4 stages of a Sierpinski triangle.
The stage 0 is an equilateral triangle.

In stage 1, it is divided into 4 congruent equilateral triangles and the middle one is removed (or shaded).

In stage 2, each of the 3 remaining triangles are divided into 4 congruent equilateral triangles and in every larger triangle, the middle one is removed (or shaded).
Further stages are obtained by repeating this process. Can you construct the first 4 stages of the Sierpinski triangle?

You can show the steps on the board as well.

Suggestions:
If possible please draw stage 0 and stage 3 on the board.

One way of drawing the triangles is to have 4 columns on the board with label indicating the stage. It will take 35 - 40 min to construct stage 0 to stage 3, so you can tell them to do it at home. And bring it next time. This will save your time for the session to proceed further. Following instruction sheet you can give them to construct Sierpinski triangle at home.

Constructing the Sierspinski Triangle.

Step 1:

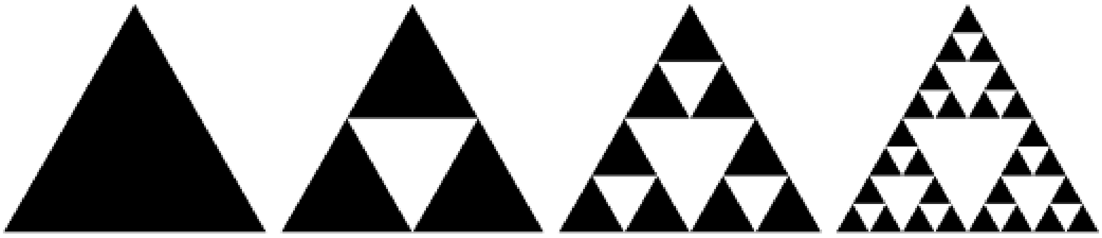
Draw an equilateral triangle. Shade or color the entire triangle. This will be called stage 0.

Step 2:

Draw another equilateral triangle of the same size as Step 1. Mark the midpoints of the three sides and join them to obtain four smaller equilateral triangles. Shade or color the three side triangles, that is, the triangles pointing upwards, leaving out the middle triangle. This figure is stage 1.

Step 3:

Construct the same figure as in stage 1 except for the shading. In each of the three triangles which were shaded in stage 1, repeat the process of joining the midpoints of the sides and shading only the triangles pointing upwards, each time, leaving out the middle triangle. This shaded figure is stage 2. The figure given below shows the Sierpinski triangle up to stage 3.



Stage 0 Stage 1 Stage 2 Stage 3

Number Patterns in the Sierpinski Triangle

Investigation 1

Count the number of shaded triangles up to stage 4 of the Sierpinski triangle. What will be the number of shaded triangles at stage 5? Complete the given table.

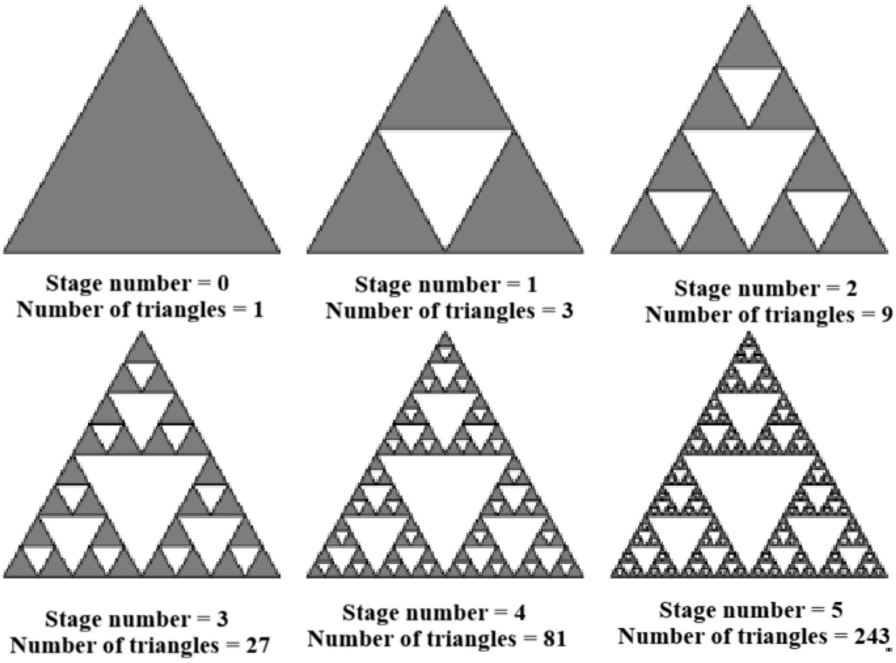
Stage	0	1	2	3	4	5
Number of shaded triangles	1					

Did you notice a pattern?

Stage	0	1	2	3	4	5	....	*n*
Number of shaded triangles	1						....	

Can you express the number of shaded triangles at any stage, in terms of the number of triangles in the previous stage?

What will happen to the number of triangles as *n* increases?



The number of shaded triangles at stage 5 of the Sierpinski triangle is 243.

The number of triangles at every stage is obtained by multiplying the number of triangles of the previous stage by 3.

The formula for the number of shaded triangles at stage *n* is given by 3^n.

Stage	0	1	2	3	4	5	......	*n*
Number of shaded triangles	1	3	9	27	81	243	......	3^n

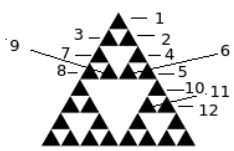
As *n* increases indefinitely, the number of shaded triangles also increases indefinitely. This needs to be discussed at length with the students.

Investigation 2:

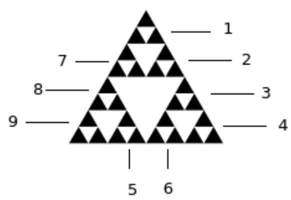
How many copies of stages 0,1 and 2 do you expect to find in stage 3 of the Sierpinski Triangle? In general how many copies of the previous stages will you find in stage n?

Finding self-similarity

Note that stage 1 of the Sierpinski Triangle comprises three identical smaller copies of stage 0 where each copy is a smaller equilateral triangle. Similarly stage 2 has three copies of



Stage 3



Stage 3

If you continue counting, you will get 27 copies of Stage 0 in Stage 3 stage 1 and nine copies of stage 0. Can you identify these?

You might have to help students find a way to count the triangles. Also remember that though there are copies of the earlier stage, the copies are of a smaller size.

Now,the number of shaded triangles at any stage, $t(n)$ and the number of triangles in the previous stage $t(n - 1)$ then $t(n) = 3 \times t(n - 1)$.
In stage 3 – number of copies of the previous stages will be –

There will be 3 copies of stage 2 in stage 3.
There will be 9 copies of stage 1 in stage 3.
There will be 27 copies of stage 0 in stage 3.

As the number of copies increases by 3^{n+1}, where n is the stage number.
We can say that, in stage n – number of copies of the previous stages will be as follows,

Copies of stage
    Stage 2 – 3^{n-2} copies
    Stage 1 – 3^{n-1} copies
    Stage 0 – 3^{n} copies and so now...

Exploring the concept of area within the Sierpinski triangle

Investigation 3

Let the area of the shaded triangle at stage 0 be A sq. unit, write down the shaded areas up to stage 4. What will be the area at stage 5?

Stage	0	1	2	3	4	5
Area of shaded triangles (in sq. units)	A					

What constant multiplier can be used to go from one stage to the next? Express the area of shaded triangles as are powers of a given fraction.

Stage	0	1	2	3	4	5	....	n
Area of shaded triangles (in sq. units)	A						....	

What happens to the area of the shaded triangles as n increases indefinitely?
Can you express the shaded area at any given stage in terms of the shaded area of the previous stage?

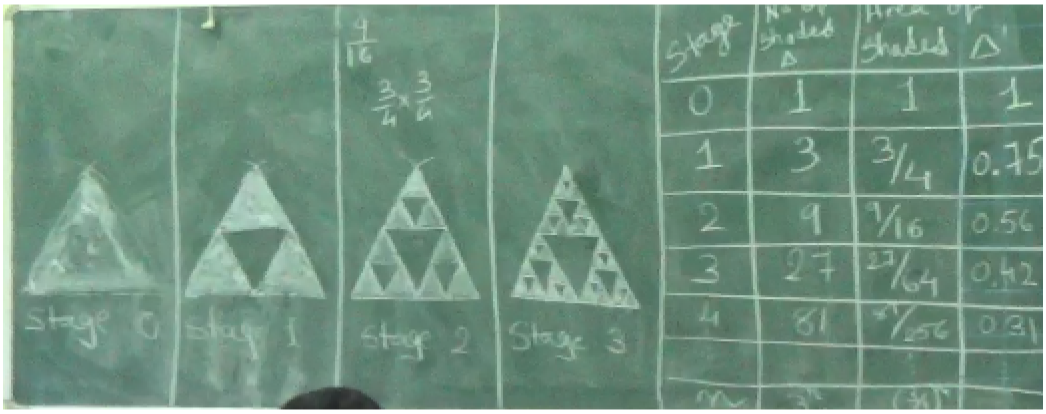
Let us suppose that the area of the shaded triangle at stage 0 is A sq. unit. At stage 1 this area is divided into four equal parts of which three parts are shaded. Thus the shaded area is $\frac{3}{4} \times A$ sq. units. At stage 2, the shaded area is $\frac{9}{16}$ sq. units. Thus the shaded area at every stage is obtained by multiplying the shaded area of the previous stage by $\frac{3}{4}$. The formula for the area of the shaded triangles at stage *n* is thus given by $\left(\frac{3}{4}\right)^n$.

Stage	0	1	2	3	4	5	………	*n*
Area of shaded triangles (in sq. units)	1	$\frac{3}{4}$	$\frac{9}{16}$	$\frac{27}{64}$	$\frac{81}{256}$	$\frac{243}{1024}$	………	$(\frac{3}{4})^n$

As *n* increases indefinitely, the area of shaded triangles decreases and approaches 0. Do spend some time on this.

Suggestions:

If possible please try to have a table just adjacent to the drawings that students have made of stage 0 – stage 3. This will help them to understand the relations between each stage also.



Also, there is a possibility that students give answers for 'Area of shaded triangles' in decimals and you will need them afterwards. So just write them down.

Exploring the concept of the length of the boundary within the Sierpinski triangle

Investigation 4

Suppose the side of the triangle at stage 0 is 1 unit. What is the length of its boundary? Find the lengths of the boundary at every stage 5?

Stage	0	1	2	3	4	5
Length of the boundary	3					

What constant multiplier can be used to go from one stage to the next?

Stage	0	1	2	3	4	5	....	*n*
Length of the boundary	3						....	

What happens to the length of the boundary of the shaded triangles as *n* increases?

Can you express the length of the boundary of the shaded triangles at any given stage in terms of the length of the boundary of the shaded triangles of the previous stage?

The length of the boundary at stage 0 is 3 units.

In stage 1, the figure consists of the outer triangle (the length of the boundary is 3 units) and the length of an inner triangle $3 \times \frac{1}{2} = \frac{3}{2}$ units. Thus the length of boundary at stage 1 is $\frac{9}{2}$ units.

At stage 2, three small triangles i.e 9 line segments of length ¼ get added. Thus the length of the boundary at this stage is $\frac{9}{2} + \frac{9}{4} = \frac{27}{4}$. We observe from the pattern that the length of a boundary at each successive stage is one and half times the length of the boundary at the earlier stage.

$P_0 = 3$

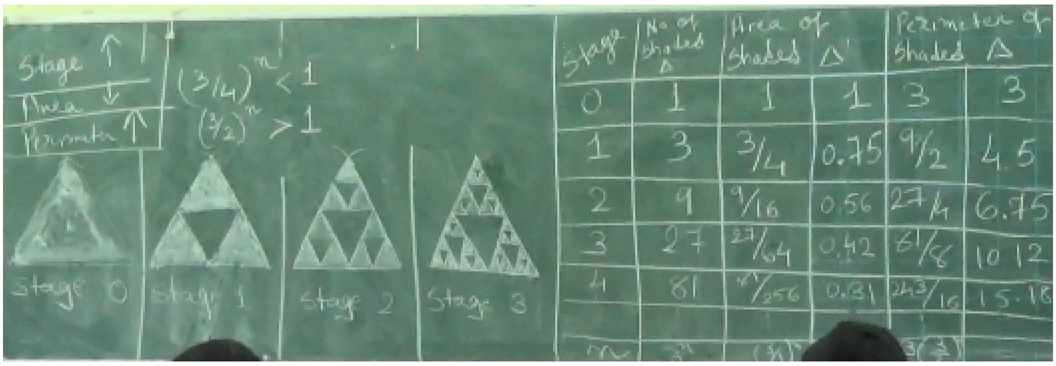
$P_1 = \frac{3}{2} \times 3,$

$P_2 = \frac{3}{2} \times \frac{3}{2} \times 3 = (\frac{3}{2})^2 \times 3$ and so on...

So, for *n*th stage the length of the boundary will be $P_n = (\frac{3}{2})^n \times 3$.

Stage	0	1	2	3	4	5
Length of the boundary (in units)	3	$\frac{9}{2}$	$\frac{27}{4}$	$\frac{81}{8}$	$\frac{243}{16}$	$\frac{729}{32}$

Suggestions:



You can also draw the length of the boundary table in the same table, as students are comfortable with the conversion of fractions into decimals and write down both these types of responses on the board.

Once everyone finishes with their tables, ask the following questions.

What do you think is happening with the area of shaded triangles when stages are increasing? What is happening with the length of the boundary when the stages are increasing?

Give some time to students to think about it. Once they arrive at some observations, ask them why?

As the number of stages increase --- area decreases
As the number of stages increases – length of the boundary increases

Let them come up with their own answers and arguments. Provoke them by asking what are the multipliers in both the cases?

In the case of area, the multiplier was $\frac{3}{4}$ which is < 1. Hence as the number of stages increases the area goes to 0.

And in the case of length of the boundary, the multiplier was $\frac{3}{2}$ so if you just take $\frac{3}{2}$ which is > 1 and as the number of stages increases the length of the boundary will go towards infinity.

A spreadsheet exploration

We shall now explore the Sierpinski Triangle construction process numerically using a spreadsheet. In Column A we shall represent the stage numbers *n*. In columns B, C and D, we shall represent the number of shaded triangles, the shaded areas at various stages and the length of the boundary at various stages respectively. The steps are as follows:

Step 1:

In column A, enter 0 in cell A2 and enter the formula = A2 + 1 in cell A3. Drag the cell A3 (till A22) to generate the stage numbers till 20. This column will represent the stage numbers *n*.

Step 2:

In column B, we enter 1 in cell B2 and the formula = B2 × 3 in cell B3. Double click the cursor in the corner of the cell B3 to generate the sequence of number of shaded triangles.

Step 3:

In column C, enter 1 in cell C2 and the formula =C2 × $(\frac{3}{4})$ in cell C3. Double clicking the cell C3 will generate the sequence of shaded areas.

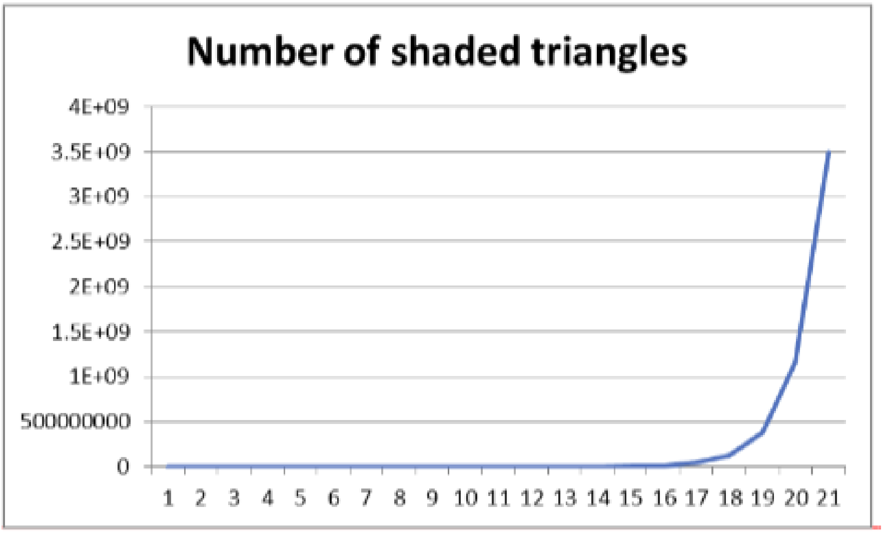
Step 4:

In column D, enter 3 in cell D2 and the formula =D2 × $(\frac{3}{2})$ in cell D3. Double clicking the cell D3 will generate the sequence of the length of the boundary at every stage.

Investigation 5:s

Generate the data on a spread sheet and observe what happens to the column values as the stages increase. Use the Graphing feature to generate the graphs for the number of triangles, shaded area and length of the boundary at each stage. What do you observe from the graphs?

The following figure shows an spreed-sheet screen-shot of the numerical exploration of the Sierpinski triangle obtained after performing steps 1 to 4. Encourage students to articulate their observations by looking at the data before obtaining the graphs.



In order to view the data graphically the following steps may be followed:

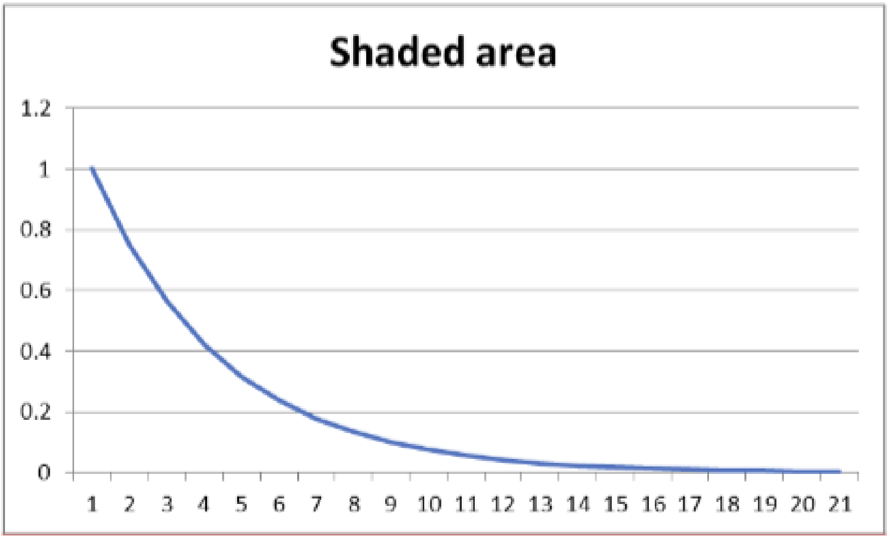
Step 1:

Select column B. Go to Insert in the tool-bar and select the line graph option. This will produce a graph where the number of stages is represented on the x-axis and the number of triangles is represented on the y-axis. The output immediately highlights the fact that the number of shaded triangles grows very quickly.

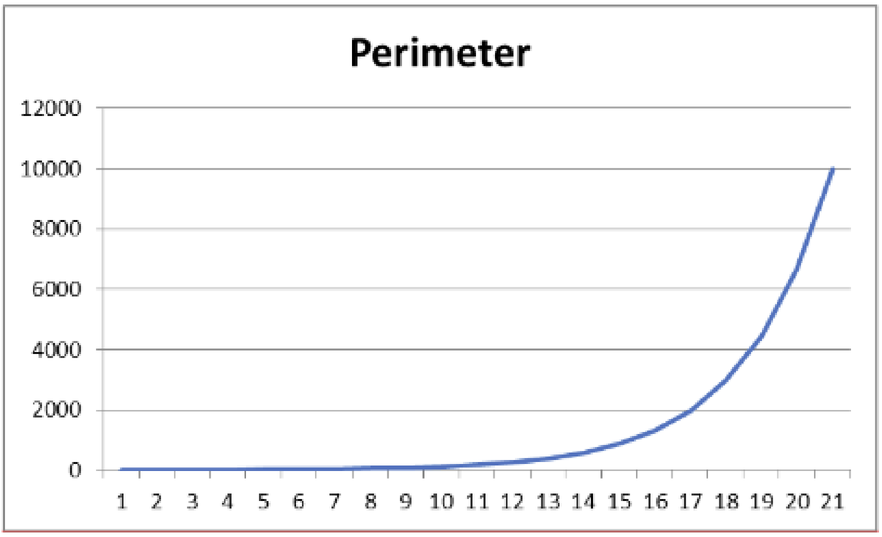
	A	B	C	D
1	Stage number	Number of shaded triangles	Shaded area	Perimeter
2	0	1	1	3
3	1	3	0.75	4.5
4	2	9	0.5625	6.75
5	3	27	0.421875	10.125
6	4	81	0.31640625	15.1875
7	5	243	0.237304688	22.78125
8	6	729	0.177978516	34.171875
9	7	2187	0.133483887	51.2578125
10	8	6561	0.100112915	76.88671875
11	9	19683	0.075084686	115.3300781
12	10	59049	0.056313515	172.9951172
13	11	177147	0.042235136	259.4926758
14	12	531441	0.031676352	389.2390137
15	13	1594323	0.023757264	583.8585205
16	14	4782969	0.017817948	875.7877808
17	15	14348907	0.013363461	1313.681671
18	16	43046721	0.010022596	1970.522507
19	17	129140163	0.007516947	2955.78376
20	18	387420489	0.00563771	4433.67564
21	19	1162261467	0.004228283	6650.51346
22	20	3486784401	0.003171212	9975.77019

Step 2:

Now select columns C and D each time producing a graph.



Note that as the number of stages increases, the shaded area approaches 0 whereas the length of the boundary increases exponentially. Ask students to imagine what the graphs would look like if the data were extended to stage 50 to 100.



Suggested readings:

https://en.wikipedia.org/wiki/Self-similarity
Book: Chaos Fractals And Self Organisation by Arvind Kumar
https://nrich.maths.org/11062
https://nrich.maths.org/4757
Kinach, Barbara M. 2014. "Generalizing: the Core of Algebraic Thinking." Mathematics Teacher 107 (6): 432–39.
Kirwan, J. Vince, and Jennifer M. Tobias. 2014. "Multiple Representations and Connections with the Sierpinski Triangle." Mathematics Teacher 107 (9): 666–71
http://fractalfoundation.org/

References :

Book: Hands-on-mathematics, class 7 by Jonaki B Ghosh (Author), Haneet Gandhi (Author), Tandeep Kaur (Author) .
Images: https://commons.wikimedia.org/wiki/Fractal
https://www.jstor.org/stable/10.5951/mathteacher.109.9.0693?seq=1#page_scan_tab_contents